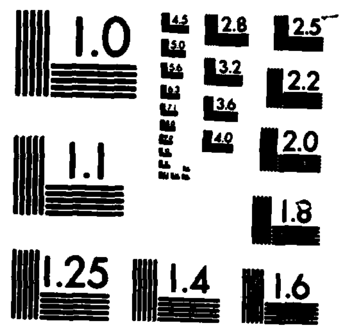
AD-A150 736 MODELING AND CONTROL OF LARGE FLEXIBLE STRUCTURES(U) 1/2
SYSTEMS ENGINEERING FOR POWER INC ANNANDALE VA
B AVRAMOVIC ET AL. 31 JUL 84 SEPI-TR-84-9
UNCLASSIFIED AFOSR-TR-85-0075 F49620-83-C-0159 F/G 12/1 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD-A150 736

AFOSR-TR- 85-0075

(3)

SEPI

SYSTEMS ENGINEERING FOR POWER, INC.
ANNANDALE, VIRGINIA

MODELING AND CONTROL OF LARGE FLEXIBLE STRUCTURES

Final Report on Phase I Research

Contract No. F49620-83-C-0159

AFOSR/NA
Directorate of Aerospace Sciences
Building 410
Bolling AFB
Washington, D.C. 20332

ATTN: Dr. A. Amos

July 31, 1984

DTIC FILE COPY

Approved for public release;
distribution unlimited.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 85-0075	2. GOVT ACCESSION NO. AD-A150736	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Modeling and Control of Large Flexible Structures		5. TYPE OF REPORT & PERIOD COVERED Final Sept. 30, 1983-May 31, 1984
		6. PERFORMING ORG. REPORT NUMBER SEPI TR 84-9
7. AUTHOR(s) B. Avramovic, N. Barkakati, W. Bennett, G.L. Blankenship, H.G. Kwatny		8. CONTRACT OR GRANT NUMBER(s) F49620-83-C-0159
9. PERFORMING ORGANIZATION NAME AND ADDRESS Systems Engineering for Power, Inc., 4300 Evergreen Lane, Suite 302, Annandale, VA 22003		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS ~~61102F~~ 3005 A1
11. CONTROLLING OFFICE NAME AND ADDRESS AFOSR/NA BOLLING AFB DC 20332-6448		12. REPORT DATE July 31, 1984
		13. NUMBER OF PAGES
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE N/A

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

from back p.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

ACTIVE CONTROLS, VIBRATION CONTROL, DISTRIBUTED FEEDBACK CONTROL, SPECTRAL METHODS, HOMOGENIZATION, LATTICE STRUCTURE, PERIODICITY,

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The main emphasis in the first phase of this work has been the adaptation and enhancement of certain Wiener-Hopf methods for control system design used by J. Davis for the treatment of linear, dynamic, distributed parameter models of flexible structures. Davis developed a frequency domain methodology for computing optimal (regulator) feedback gains for linear distributed parameter control systems by Wiener-Hopf spectral factorization. The numerical algorithms for executing the spectral factorization were based on some earlier work of F. Stenger. over

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

5

The Davis-Stenger methodology was adapted to the problem of vibration control of flexible structures. The spectral factorization methodology avoids the difficult numerical problems associated with the solution of the Riccati partial differential equations which arise in the time domain approach for designing stabilizing controllers. In this way distributed phenomena, like travelling waves, which characterize the macroscopic dynamics of flexible structures are retained in the model, and their interaction with the control system is preserved in the analytical design process. Computational algorithms were developed and several prototype systems were treated including the Euler Beam and a simple two dimensional system.

Second part of the research involved the use of a mathematical technique for asymptotic analysis called "homogenization," originally developed by I. Babuska, to produce simplified models for flexible structures with a regular (periodic) infrastructure. Homogenization of the model for a structure with a regular infrastructure produces a model with smoothly varying "effective" parameters for mass density, local tension, and damping that represents a flexible structure with a uniform "homogenized" internal structure. Originator-supplied keywords include: → front p

The homogenization technique does not require the a priori assumption of a specific continuum structure as the approximation for a given lattice structure. Instead, the asymptotic analysis of the original structure produces the distributed continuum approximation model of the lattice structure in the limit as some characteristic parameter (e.g., inter-cell dimension) in the structural model goes to zero. Moreover, the natural averaging process is developed in the course of the analysis. It is easy to construct examples in which the procedure of averaging parameters over a characteristic volume leads to incorrect approximations for the system dynamics. The homogenization methods used in this research are based on the assumption of a periodic infrastructure in the original model. This is not necessary, and random structures can also be treated, if the randomness has sufficient ergodicity properties (in the spatial variables).

Homogenization and asymptotic analysis can also be carried out in the context of control and state estimation problems for heterogeneous structures. It is important that the control and homogenization procedure not be done separately, since one can construct examples in which control designs based on averaged models are not correct as approximations to the optimal (e.g., regulator) control laws for the original problem. While control and filtering theory with homogenization is not very advanced at this stage, prototype problems can be analyzed to a point where the basic features of design algorithms are clear.

MODELING AND CONTROL OF LARGE FLEXIBLE STRUCTURES

Final Report on Phase I Research

Contract No: F49620-83-C-0159

AFOSR/NA
Directorate of Aerospace Sciences
Building 410
Bolling AFB
Washington, DC 20332

ATTN: Dr. A. Amos

31 July 1984

Submitted by:

SEPI
Suite 302
4300 Evergreen Lane
Annandale, VA 22003
(703)-941-6206

Prepared by:

B. Avramovic
N. Barkakati
W. Bennett
G.L. Blankenship
H.G. Kwatny

CONTENTS

List of Figures

Summary of Phase I Research

Interaction of the control system with the structural dynamics of the physical system is one of the fundamental issues in large space structure applications. Our work is intended to contribute directly to understanding this interaction by using models which capture the essential distributed character of the system, and using analytical techniques which preserve the character of the physical system in the model simplification process. The methods we have used -- Wiener-Hopf/spectral factorization methods for design of distributed control systems and homogenization/asymptotic analysis for model simplification -- have tremendous potential for the analytical treatment of complex structural control problems, including the sysnthesis of computer-aided-design methods for large space structures. In Phase I of this project, we have concentrated on the treatment of a few simple prototype systems. Further work is needed to adapt and enhance the methods to treat complex structures. The analytical methods themselves do not require substantial extensions. Rather, their potential for the treatment of complex flexible structures should be developed.

The main emphasis in the first phase of this work has been the adaptation and enhancement of certain Wiener-Hopf methods for control system design used by J. Davis for the treatment of linear, dynamic, distributed parameter models of flexible structures. Davis developed a frequency domain methodology for computing optimal (regulator) feedback gains for linear distributed parameter control systems by Wiener-Hopf spectral factorization. The numerical algorithms for executing the spectral factorization were based on some earlier work of F. Stenger. We have adapted the Davis-Stenger methodology to the problem of vibration control of flexible structures. A generic problem of this type is the figure control of a large space antenna. We have carried out the analysis and computed the optimal feedback regulator control laws for several examples including the Euler-Bernoulli beam model and a two-dimensional prototype (experimental) system studied by J. Lang and D. Staelin.

This portion of the research has demonstrated the effectiveness of frequency domain -- spectral factorization methods for the design of control and state estimation algorithms for flexible structures described by linear distributed parameter models (hyperbolic partial differential equations). In this approach it is not necessary to reduce the models to finite dimensional (lumped parameter) models at the outset of the design procedure. The infinite dimensional character of the system is preserved throughout the design process. The spectral factorization methodology avoids the difficult numerical problems associated with the solution of the Riccati partial differential equations which arise in the time domain approach for designing stabilizing controllers. In this way distributed phenomena, like travelling waves, which characterize the macroscopic dynamics of flexible structures are retained in the model, and their interaction with the control system is preserved in the analytical design process.

In the second part of the research we have examined the use of a mathematical technique for asymptotic analysis called "homogenization", originally developed by I. Babuska, to produce simplified models for flexible structures with a regular (periodic) infrastructure. Homogenization of the model for a structure with a regular infrastructure produces a model with smoothly varying "effective" parameters for mass density, local tension, and damping that represents a flexible structure with a uniform "homogenized" internal structure. The derivation of continuum models for complex structures with a regular infrastructure has been studied for many years in applied mechanics. In most cases the continuum models are based on local averages of the physical parameters (e.g., mass density, stress, strain, etc.) over some characteristic volume of the structure. The averaged quantities computed in this way are related to the associated quantities in a postulated continuum structure. For example, the mass density and stress tensors in a long truss with a regular lattice structure have been related to the distributed parameters in a beam (in the work of Noor, Nayfeh, and Renton, among others).

Our technique does not require the a priori assumption of a specific continuum structure as the approximation for a given lattice structure. Instead, the asymptotic analysis of the original structure produces the distributed continuum approximation model of the lattice structure in the limit as some characteristic parameter (e.g., inter-cell dimension) in the structural model goes to zero. Moreover, the natural averaging process is developed in the course of the analysis. It is easy to construct examples in which the usual procedure of averaging parameters over a characteristic volume leads to incorrect approximations for the system dynamics. The homogenization methods used in our research are based on the assumption of a periodic infrastructure in the original model. This is not necessary, and random structures can also be treated, if the randomness has sufficient ergodicity properties (in the spatial variables). Numerical evaluations of the averaged model are more difficult in this case.

Homogenization and asymptotic analysis can also be carried out in the context of control and state estimation problems for heterogeneous structures. It is important that the control and homogenization procedure not be done separately, since one can construct examples in which control designs based on averaged models are not correct as approximations to the optimal (e.g., regulator) control laws for the original problem. While control and filtering theory with homogenization is not very advanced at this stage, it is nevertheless possible to analyze some prototype problems to a point where the basic features of the theory are clear. Our work has contributed to this process, but much more needs to be done.

Part I:

Wiener-Hopf Methods for Design of Stabilizing Control Systems

1. Background: Dynamical Control of Flexible Structures

Interaction of the control system with the structural dynamics of the mechanical system is one of the fundamental issues in large space structure applications. Our work is intended to contribute directly to understanding this interaction by using models which capture the essential distributed character of the system, and using analytical techniques which preserve the character of the physical system in the model simplification process. The methods we have used -- Wiener-Hopf/spectral factorization methods for design of distributed control systems and homogenization/asymptotic analysis for model simplification -- have tremendous potential for the analytical treatment of complex structural control problems, including the synthesis of computer-aided-design methods for large space structures. In Phase I of this project, we have concentrated on the treatment of a few simple prototype systems. The methods may be adapted to treat complex structures. They do not require substantial extensions for such cases. Rather, computational algorithms which translate their strengths into effective design tools need to be developed.

The main emphasis in the first part of this work has been the adaptation and enhancement of certain Wiener-Hopf methods for control system design used by J. Davis for the treatment of linear, dynamic, distributed parameter models of flexible structures (Davis 1978, 1979a,b 1982) (Davis and Barry 1977) (Davis and Dickenson 1983). Davis and his colleagues developed a frequency domain methodology for computing optimal (regulator) feedback gains for linear distributed parameter control systems by Wiener-Hopf spectral factorization. The numerical algorithms for executing the spectral factorization were based on some earlier work of F. Stenger (1972). We have adapted the Davis-Stenger methodology to the problem of vibration control of flexible structures. A generic problem of this type is the figure control of a large space antenna. We have carried out the analysis and computed the optimal feedback regulator control laws for several examples including the Euler-beam and a two-dimensional prototype (experimental) system studied by J. Lang and D. Staelin (Lang and Staelin 1982a,b).

This portion of the research has demonstrated the effectiveness of frequency domain -- spectral factorization methods for the design of control and state estimation algorithms for flexible structures described by linear distributed parameter models (hyperbolic partial differential equations). In this approach it is not necessary to reduce the models to finite dimensional (lumped parameter) models at the outset of the design procedure. The infinite dimensional character of the system is preserved throughout. The spectral factorization methodology avoids the difficult numerical problems associated with the solution of the Riccati partial differential equations which arise in the time domain approach for designing stabilizing controllers. In this way distributed phenomena, like travelling waves, which characterize the macroscopic dynamics of flexible structures are retained in the model, and their interaction with the control system is preserved in the analytical design process.

1.1 Generic Models of the Dynamics of Flexible Structures

The flexible structures treated in this work are assumed to be continua described generically by the system of partial differential equations

$$(1) \qquad m(x)h_{tt}(t,x) + D_0 h_t(t,x) + A_0 h(t,x) = F(t,x)$$

where $h(t,x)$ is an n-vector of instantaneous displacements away from its equilibrium of the structure S, a bounded open set in R^n with smooth boundary S. The mass density $m(x)$ is positive and bounded on S. The damping term $D_0 h_t$ contain both (asymmetric) gyroscopic and (asymmetric) structural damping effects. The internal restoring force term $A_0 h$ is generated by a time-invariant differential operator A_0 specific to the flexible structure. For most cases of interest, A_0 may be taken to be an unbounded operator with domain $D(A_0)$ containing smooth functions with the appropriate boundary conditions which is dense in the Hilbert space $H_0 = L^2(S)$ with the natural inner product, $\langle \cdot,\cdot \rangle$. In many cases A_0 has a discrete spectrum with associated eigenfunctions which constitute a basis for $L^2(S)$.

The applied force distribution $F(t,x)$ generally has three components

$$(2) \qquad F(t,x) = F_d(t,x) + F_c(t,x) + F_a(t,x)$$

where $F_d(t,x)$ is a vector of exogeneous disturbance forces and torques, $F_c(t,x)$ is a continuous, distributed, controlled force field (as in an electrostatically controlled system); and $F_a(t,x)$

represents the control forces due to discrete actuators

$$(3) \qquad F_a(t,x) = \sum_{j=1}^{m} b_j(x)u_j(t) \triangleq B_0u(t)$$

The actuator amplitudes are u (t) and the actuator influence functions $b_j(x)$ are typically elements of H_0 (which usually, but not always, approximate delta functions $\delta(x-x_0)$). Observations are usually assumed to arise from a finite number p of sensors

$$(4) \qquad y_i(t) = \langle c_j,h\rangle_0 + \langle \tilde{c}_j,h_t\rangle_0 \quad j=1,\ldots,p$$

or

$$(4') \qquad y(t) \triangleq C_0h + \tilde{C}_0h_t \; ; \; y(t) = \left[y_1(t),\ldots,y_p(t)\right]$$

where the position and velocity influence functions c_j, $\tilde{c}_j$, $j = 1,\ldots,p$ are elements of H which may represent point devices. Note that $B_0:R^n \to H$, $C_0:R^p \to H_0$, and $\tilde{C}_0:R \to H$ are bounded.

The control problem for (1)-(4) is to choose the discrete control amplitudes $u_j(t)$, $j = 1,\ldots,m$, and the distributed control forces $F_c(t,x)$, based on the observations $y_i(t)$, $i = 1,\ldots,p$ to maintain the state

$$(5) \qquad v(t) = \begin{bmatrix} h(t,x) \\ h_t(t,x) \end{bmatrix}$$

as close to its equilibrium position (nominally zero) as possible. If the disturbances are transient, this may be accomplished by using a regulator control law which minimizes the quadratic performance index

(6) $J(u) = \int_0^\infty [q(v,v) + \alpha u^T(t)u(t) + Q(F_c, F_c)]dt$

where q, Q are non-negative quadratic forms, and α is a positive parameter. This is the generic control problem surveyed in (Balas 1982). It includes boundary and interior control of vibrating strings, membranes, thin beams, and thin-plates.

1.2 State space models and modal control

Suppose for the moment that A_0 is symmetric with compact resolvent and spectrum bounded from below. The spectrum of A_0 consists, therefore, of isolated eigenvalues λ_k,

(7) $\lambda_1 \le \lambda_2 \le \ \ldots$

and eigenvectors such that $A_0\phi_k = \lambda_k\phi_k$. Assume $\lambda_1 > 0$. Then A_0 satisfies

(8) $\langle A_0 h, h\rangle_0 \ge \varepsilon||h||_0^2, \varepsilon > 0$

and A_0 has a square root $A_0^{\frac{1}{2}}$. Let $D(A_0) \subset L^2(S)$ be the domain of A_0, and $D(A_0^{\frac{1}{2}}) \subset L^2(S)$ be that of $A_0^{\frac{1}{2}}$. Let $H = L^2(S) \times L^2(S)$, and consider

(9) $A \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} w \\ -A_0 v \end{bmatrix}$, $v \in D(A_0)$, $w \in D(A_0^{\frac{1}{2}})$

(10) $B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}$ $\qquad C = [C_0 \ \ \tilde{C}_0]$

so that $B: R^n \times R^n \to H$ and $C: H \to R^P$. With v(t) defined by (5) we have

(11) $\frac{d}{dt} v(t) = Av(t) + Bu(t)$, $y(t) = Cv(t)$, $v(0) \in H_1$

which is the state space description of the original problem with the additional assumptions on A.

The energy inner product $\langle\cdot,\cdot\rangle$ defined on H_1 is

$$(12) \qquad \left\langle \begin{bmatrix} v_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ w_2 \end{bmatrix} \right\rangle_E = \langle v_1, A_0 v_2\rangle_0 + \langle w_1, w_2\rangle$$

And so, in the energy norm we have

$$(13) \qquad ||v(t)||_E = \langle h, A_0 h\rangle_0 + \langle h_t, h_t\rangle$$

which is a measure of the total potential and kinetic energy in (h, h_t). The operator A on $(H_1, \langle\cdot,\cdot\rangle_E)$ generates a unitary group U(t) (Treves, 1975) and

$$(14) \qquad U(t)v_0 = \sum_{k=1}^{\infty} \begin{bmatrix} \cos \omega_k t & \omega_k^{-1} \sin \omega_k t \\ -\omega_k \sin \omega_k t & \cos \omega_k t \end{bmatrix} \begin{bmatrix} a_k(0) \\ \dot{a}_k(0) \end{bmatrix} \phi_k$$

for any

$$(15) \qquad v_0 = \sum_{k=1}^{\infty} \begin{bmatrix} a_k(0) \\ \dot{a}_k(0) \end{bmatrix} \phi_k \in H_1$$

Thus, when $u(t) \equiv 0$ in (11), energy is preserved, i.e., $||U(t)v_0|| = ||v_0||_E$, for any $v_0 \in H$. For any u(t), continuously differentiable, the solution of (11) is

$$(16) \qquad v(t) = U(t)v_0 + \int_0^t U(t-\tau)Bu(\tau)d\tau$$

In fact,

$$(17) \quad v(t) = \sum_{k=1}^{\infty} \begin{bmatrix} a_k(t) \\ \dot{a}_k(t) \end{bmatrix} \phi_k$$

with $[a_k(t), \dot{a}_k(t)]$, $k = 1,2,\ldots,$ defined by ordinary differential equations.

By introducing finite dimensional subspaces $H^k = \text{span}\{\phi_k, k = 1,2,\ldots,K\}$ of H_0, one can construct finite dimensional modal approximations to the system (11); and from these, finite dimensional control problems whose solutions may be used to compute suboptimal control laws for the infinite dimensional control problem defined by (6)(11). (See, e.g., Balas 1978, 1982). The feedback controls obtained in this way will stabilize the first K modes of the distributed system. However, as noted in (Balas 1978) in all but a few special systems, the control actions will excite the higher order modes. This "spillover" effect invariably degrades system performance. While this phenomenon has received considerable attention in the literature, it is the unavoidable companion of design methods based on lumped parameter models.

In this research we take a different approach to the control system design which deals directly with the infinite dimensional system. The method uses a frequency domain formulation of the control problem, analogous to the setup for finite dimensional problems in (Willems 1971), and a spectral factorization algorithm to compute the feedback gain. The formal algorithms are described in section 2. Before developing the mathematics it is useful to look at some examples and prototype systems which illustrate the basic features of the control problem.

1.3 Control of a vibrating flexible string

Small vibrations of a flexible string may be described by

$$(18) \quad \rho(z)h_{tt} - p(z)h_{zz} + q(z)h_t + r(z)h_z = 0$$

where $\rho(z) \geq \rho_0 > 0$ is the linear mass density, $p(z) > p_0 > 0$ is the modulus of elasticity, and we assume that ρ, p, q, r are twice continuously differentiable. Suppose that the space $z \varepsilon [0,1]$ and time t have been normalized to dimesionless coordinates. The system can be put in a standard form by changing the independent variable

$$x = \int_0^z \left(\frac{\rho(s)}{p(s)} \right)^{\frac{1}{2}} ds$$

with L = x(1) we have

$$(19) \quad \begin{array}{c} h_{tt} - h_{xx} + a(x)h_t + b(x)\, h_x = 0 \\ 0 \leq x \leq L \; , \; 0 \leq t \end{array}$$

The coefficients a(x) and b(x) are continuously differentiable functions of x. Defining

$$(20) \qquad v = h_t(t,x), \qquad w = h_x(t,x)$$

we have

$$(21) \quad \frac{\partial}{\partial t}\begin{bmatrix} v \\ w \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} v \\ w \end{bmatrix} + \begin{bmatrix} a(x) & b(x) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = 0$$

The appropriate boundary conditions are

$$(22) \quad \begin{array}{l} \alpha_0 v(t,0) + \beta_0 w(t,0) = 0 \\ \alpha_1 v(t,L) + \beta_1 w(t,L) = u(t) \end{array}$$

with the conditions $(\alpha_0/\beta_0) \neq \pm 1$, $(\alpha_1/\beta_1) \neq \pm 1$ imposed to avoid pathologies. Here $\beta_0 = 0$ corresponds to a fixed endpoint, while $= 0$ permits an end to move freely along the h axis, and $\alpha_0 \neq 0$, $\beta_0 \neq 0$ represents an end free to move but with positive or negative friction. The function u(t) is a boundary control.

One can generalize (21) to

$$(23) \quad \frac{\partial}{\partial t} \begin{bmatrix} v \\ w \end{bmatrix} - A \begin{bmatrix} v \\ w \end{bmatrix} = 0$$

with

$$(24) \quad Af = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial x} f - \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix} f$$

and the real coefficients $a_{ij}(x)$ are continuously differentiable on $0 \leq x \leq L$. By studying the finite time controllability of (22)-(24) D.L. Russell (1972) was able to prove some interesting properties of the eigenvalues and eigenfunctions of A. In particular, if the (complex) eigenvalues of A are $\{\lambda_k\}$, then $\{ e^{\lambda_k x}, k = 1,2 \ldots\}$ forms a Riesz basis for the space $L^2[0, 2L]$. Moreover, there is a unique control $u \in L^2[0,T]$ T=2L (recall that all variables are dimensionless) which takes the solution of (22)-(24) to zero at t=T=2L from arbitrary initial conditions (in the space L^2)

$$(25) \quad v(0,x) = v_0(x), \; w(0,x) = w_0(x), \; 0 \leq x \leq L$$

and

$$(26)\quad k\int_0^L |v_0(x)|^2 + |w_0(x)|^2 dx \le \int_0^T |u(t)|^2 dt\Big|_{T=2L}$$

$$\le K\int_0^L |v_0(x)|^2 + |u_0(x)|^2\, dx$$

for some positive constants k, K. The time T = 2L is "critical". In general, it is not possible to make the transfer in $T < 2L$; and for $T > 2L$ there will, in general, be many controls which accomplish the transfer. By considering the special spectral structure of the operator A and its adjoint A^*, Russell was able to show that the unique control $u(t)$, $0 \le t \le 2L$, accomplishing the transfer could be synthesized by a bounded linear functional of the state in (21).

From this analysis it follows that the optimal regulator problem for (21), that is, the problem

$$(27)\quad \min_{u \in U_{ad}} \int_0^\infty [u^2(t) + (\int_0^L |v(t,x)|^2 + |w(t,x)|^2 dx)]dt$$

subject to (21) (22) with admissible controls consisting of bounded (linear) feedback functionals of the state has a unique solution which produces a finite optimal cost. The problem (23)-(25) (27) is the simplest example of the class of control problems treated in this work. It is a one dimensional version of the two dimensional prototype system discussed next.

1.4 Control of a two-dimensional hyperbolic system

In an interesting paper J. Lang and D. Staelin (1982b) studied the dynamical control properties of a simple experimental system as a prototype of an antenna design using electrostatic control to maintain the antenna shape (Lang and Staelin 1982a). The experimental system consisted of a flexible, conducting wire mesh (about 1 m^2) suspended vertically in tension by rigid rectangular boundaries and biased by a high voltage source. A parallel surface of equal dimension, spaced a short distance normal to the mesh, supported a 3 x 3 array of fixed conducting plates independently addressable through bipolar, variable low voltage sources which collectively served as a distributed, electrostatic control. A similar set of plates, equally spaced from the mesh on the opposite side, served to capacitively sense mesh deflections. The balancing electrostatic pressures on the mesh produced a grounded-control equilibrium geometry in which all three surfaces were parallel.

A regulator control law was designed to modulate the voltages on the 9 actuator plates in response to (filtered) observations of the mesh deflections from the 9 sensor plates. Finite dimensional modal models representing the dynamics of 1 to 3 modes were used in the control system design. The basic linear - quadratic - Gaussian regulator control law was not satisfactory in certain experimental regimes (high bias voltage) due to unmodeled physical factors. Modifications were necessary to achieve mesh stabilization in these cases. Spillover effects were also observed and compensated.

In (Lang and Staelin 1982b) the mesh was modeled as a flexible membrane in tension. The transverse mesh deflection h(t,x,y), defined as positive toward the sensor plates, satisfies

$$(28)\qquad M h_{tt} = T_\alpha h_{xx} + T_\beta h_{yy} - D h_t + f \quad .$$

Here M is (uniform) mass density, T_α, T_β are (uniform) coefficients of mesh tension, and D is a viscous damping coefficient. Assuming a long wave model for the electric field between the mesh and plates, the net transverse electrostatic pressure, f(t,y,y), acting on the mesh satisfies

$$(29) \quad f = \tfrac{1}{2} \varepsilon_o \left[\frac{V^2}{(H-h)} - \frac{(V-u)^2}{(H+h)} \right],$$

where u = u(t,x,y) is the potential of the actuator plates, V is the mesh bias potential, H is the electromechanical plate-to-mesh separation, and ε_0 is the permittivity of free space. Assuming

$$|h| << H \text{ and } |u| << V$$

equation (29) was linearized, and the resulting linear control system studied in (Lang and Staelin 1982b) was

$$(30) \quad Mh_{tt} = T_\alpha h_{xx} + T_\beta h_{yy} - Dh_t + Kh - Bu(t)$$

where $K = 2\varepsilon_0 V^2/H^3$ and $B = \varepsilon_0 V/H^2$. Defining $S = [0, L_\alpha] \times [0, L_\beta]$ as the location of the mesh, the mesh boundary conditions required zero deflection at the perimeter.

If we identify Ah as the linear operator on the right in (30), then the eigenfunctions of A are

$$(31) \quad \phi_{m,n}(x,y) = \sin(m\pi x/L_\alpha) \cdot \sin(n\pi y/L_\beta)$$
$$0 \le x \le L_\alpha, \quad 0 \le y \le L_\beta$$

and the corresponding eigenvalues are

$$(32)\quad \lambda_{m,n} = -\frac{1}{2}\frac{D}{M} \pm \sqrt{\frac{D^2}{4M^2} + \frac{K}{M} - \frac{m^2\pi T_\alpha}{ML_\alpha^2} - \frac{n^2\pi T_\beta}{ML_\beta^2}}$$

These define the "open-loop" natural frequencies of the system. Notice that the (m, n)-mode is unstable if

$$(33)\quad V^2 > \frac{\pi^2 H^3}{2\,\varepsilon_0}\left[\frac{m^2 T_\alpha}{L_\alpha^2} + \frac{n^2 T_\beta}{L_\beta^2}\right]$$

Therefore, if V is large enough, a finite number of modes are open-loop unstable.

The experimental system in (Lang and Staelin 1982b) has noise in both actuator and sensor systems. This noise was represented by Gaussian white noise. The overall performance index used in the design was

$$(34)\quad J = \frac{1}{2}\lim_{k} E\left\{\frac{1}{\tau L_\alpha L_\beta}\int_{k\tau}^{(k+1)\tau}\left[\int_0^{L_\alpha}\int_0^{L_\beta} q^2h^2(t,x,y) + r^2v^2(t,x,y)\;dx\,dy\right]dt\right\}$$

where $r = v_0^{-1}$, with v_0 the dynamic range on the voltage control system, and $q = 2\ V/Hv_0$.

The study of this small system provides a great deal of useful information on control problems that can be expected for certain classes of flexible structures. The control system performance reported by Lang and Staelin provides one of the few available benchmarks against which alternative control system designs may be tested. We shall consider this system further in section 4.

1.5 Control of a simply supported beam

The Euler-Bernoulli equation for the dynamics of a simply supported beam is

$$
(35)\qquad
\begin{gathered}
Mh_{tt} + Dh_{txx} + EIh_{xxxx} = f(t,x) \\
0 \le x \le L \;,\; 0 \le t \\
h(t,0) = 0 = h(t,L) \\
h_{xx}(t,0) = 0 = h_{xx}(t,L)
\end{gathered}
$$

where M is the mass density (per unit length), D is a damping ratio, E is the modulus of elasticity, I is a moment of inertia, and f is an applied force distribution.

If we ignore the damping, D = 0, and normalize other parameters to unity, then the mode shapes - eigenfunctions are $\Phi_k(x) = \sin k\pi x$ and the eigenvalues are $\omega_k = (k\pi)^2$. Control of vibrations of the beam, may be based on the performance index

$$
(36)\quad J(u) = \int_0^\infty [\int_0^L f^2(t,x) + q_1^2 h^2(t,x) + q_2^2 h_t^2(t,x)dx)]\, dt
$$

Numerical studies of this problem were reported in (Balas 1978). A point actuator and a point sensor

$$
(37)\qquad
\begin{aligned}
f(t,x) &= u(t)\,\delta(x - \hat{x}) \\
y(t) &= h(t,\tilde{x})\,\delta(x - \hat{x})
\end{aligned}
$$

were used to effect control in the problem. Spillover into the uncontrolled residual modes produced instability in the simulations, due in part to the absence of damping.

This problem is considered in more detail in section 3.

2. Wiener-Hopf Methods for Control System Design: The Davis-Stenger Algorithm

The connections among least squares optimization, spectral factorization and algebraic Riccati equations have been considered important in control theory for many years. (See, e.g., Anderson (1967), Brockett (1970), Willems (1971), Molinari (1973a,b), Helton (1976), and the references therein.) To see how the connection arises, consider the standard finite dimensional, infinite time linear regulator problem

$$\min_{u} \int_0^\infty |u(t)|^2 + |y(t)|^2 dt$$

(1) $$\dot{x}(t) = Ax(t) + Bu(t) , x(0) = x_0$$

$$y(t) = Cx(t) , t \geq 0$$

Suppose A is a stable matrix and (A,B,C) is a minimal finite dimensional triple realizing the transfer function

(2) $G(s) = C(Is - A)^{-1}B$

Then the optimal feedback control for (1) is given by

(3) $u(t) = -B^*Kx(t)$

with K the unique positive definite symmetric solution to the algebraic Riccati equation

(4) $$A^*K + K(A - BB^*K) = -C^*C$$

An integral equation for the optimal feedback gain may be derived from (4). Let s be a complex number not in the spectrum of $-A^*$, $\sigma(-A^*)$, nor in the spectrum of $A-BB^*K$, then

$$(5) \quad K[Is - (A-BB^*K)]^{-1} + (-Is - A^*)^{-1}K = (-Is - A^*)^{-1}C^*C[-Is - (A-BB^*K)]^{-1}$$

From standard results (Brockett 1970), $\sigma(A-BB^*K)$ is contained in the open left half of the complex plane; and, by assumption, $\sigma(-A^*)$ is in the open right half plane. Let Γ be a closed rectifiable contour encircling $\sigma(A-BB^*K)$ in the positive sense, and integrate (5) along Γ. Since

$$(6) \quad \begin{aligned} \frac{1}{2\pi i}\int_\Gamma [Is - (A - BB^*K)]^{-1} ds &= I \\ \frac{1}{2\pi i}\int_\Gamma [-Is - A^*]^{-1} ds &= 0 \end{aligned}$$

we obtain

$$(7) \quad K = \frac{1}{2\pi i}\int_\Gamma [-Is - A^*]^{-1} C^*C\,[Is-(A-BB^*K)]^{-1} ds$$

and so

$$(8) \quad KB = \frac{1}{2\pi i}\int_\Gamma [\cdot Is - A^*]^{-1} C^*C\,[Is-(A-BB^*K)]^{-1} B ds$$

Since the integrand is the product of two rational functions, the contour may be deformed to yield

$$\text{(9)}\quad KB = \frac{1}{2\pi}\int_{-\infty}^{\infty}[-Ii\omega - A^*]^{-1}C^*C\,[Ii\omega - (A-BB^*K)]^{-1}B\,d\omega$$

The spectral factorization identity (Brockett 1970, Willems 1971)

$$\begin{aligned} H(i\omega) &= I + G^*(i\omega)\,G(i\omega) \\ \text{(10)}\qquad &= I + B^*[-Ii\omega - A^*]^{-1}C^*C\,[Ii\omega - A]^{-1}B \\ &= F^*(i\omega)\,F(i\omega) \\ &= [I + B^*(-Ii\omega - A^*)^{-1}KB]\,[I + B^*K\,(Ii\omega - A)^{-1}B] \end{aligned}$$

and the identity

$$\text{(11)}\quad C\,[Ii\omega - (A - BB^*K)]^{-1}B = C\,(Ii\omega - A)^{-1}B[I + B^*K\,(Ii\omega - A)^{-1}B]^{-1}$$

when used in (9) gives the result

$$\text{(12)}\quad B^*K = \frac{1}{2\pi}\int_{-\infty}^{\infty}[F^*(i\omega)]^{-1}B^*R^*(i\omega,A)C^*CR(i\omega,A)\,d\omega$$

where $R(s,A) = [Is-A]^{-1}$ is the resolvent of A.

Hence, to compute the optimal feedback gain, we can either solve the nonlinear algebraic Riccati equation (4) for its unique positive definite solution or we can carry out the spectral factorization of $I+G^*G$ in (10) and then compute the integral (12). In finite dimensional control problems there may be little reason to favor formulation of the computational problem in one setting - the Riccati equation - over the other - spectral factorization. In infinite dimensional problems, however, the spectral factorization method appears to have superior numerical stability properties over direct integration of the Riccati equation.*

2.1 Davis's Method

In a series of papers, J. Davis and his students (Davis and Barry (1977), Davis (1978a,b) (1979) (1982), Davis and Dickinson (1983)) have explored the application of spectral factorization methods for control system design to a class of distributed parameter models of long trains with multiple locomotives. The control problem is to modulate the acceleration of individual locomotives to minimize deviations in coupler stress throughout the train. Disturbances include passage of the train over a grade, which tends to set up a "travelling wave" along the train of stress deviations from nominal. The first approach to this problem which comes to mind is to write out the equations of motion of the cars and locomotives in the train and formulate an optimal control problem for the overall system. The large dimension of the resulting model and the absence of any special structure inhibits this approach. Decentralized control schemes (McLane, Peppard, and Sundareswaran (1973), Gruber and Bayoumi (1982)) are not particularly effective for these problems. As Davis and Barry (1977) have observed, aside from the difficulties in solving large scale control problems, one has trouble estimating the effects of system parameter changes or of variations in the number of units in a block based on lumped parameter models.

Davis recognized that the mass-spring nature of the interconnected system could lead to traveling wave phenomena setup by "competing" local controllers (locomotive accelerations). He reasoned that the macrosopic, widely coupled motions of the elements would contain the bulk of the energy of motion of the total system. This hypothesis suggests that a control scheme designed to suppress such motions would achieve substantial reductions in the coupler stress levels.

To represent the system in a fashion which would capture wave phenomena most naturally, Davis reformulated the system as a distributed parameter system with boundary controls. (Davis and Barry 1977). The resulting model proved to be mathematically tractable. The effects of changes in both system parameters and the number of units in a block were readily apparent. In fact, an increase in the number of elements in a block increases the validity and usefulness of the model. In contrast, finite dimensional models tend to become increasingly intractable as the number of units in the system increases.

Davis' modeling technique is simple and instructive. Consider the mass - spring - damper system in Figure 2.1. The dynamics are

$$(13) \quad m \frac{d^2}{dt^2} x_i = -K (x_{i-1} + 2x_i - x_{i+1}) - c \frac{d}{dt} (x_{i-1} + 2x_i - x_{i+1})$$

where $x_i(t)$ is the deviation of the i^{th} mass from its nominal position. The continuum approximation to this system may be developed as follows: Let $z \in [0,1]$ be spatial position along the "rest length", unity, of the overall system, and let $u(z,t)$ be the deviation of the mass at rest position z and time t. Making the identification

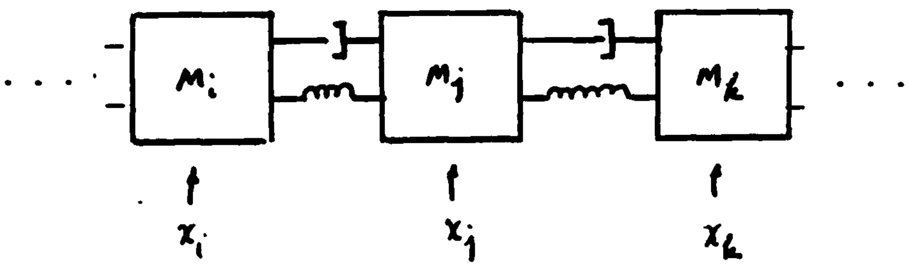


Figure 2.1 Discrete element model for viscoelastic bar

(14) $u(i/N,t) \sim X_i(t),\ i = 0,1,\ldots,$

and using the numerical differentiation formula

(15) $$\frac{1}{(1/N)^2}\left[u(z+\tfrac{1}{N}) - 2u(z) + u(z-\tfrac{1}{N})\right] \cong \frac{\partial^2 u}{\partial z^2}(z)$$

One arrives at the distributed model equation

(16) $$m\,\frac{\partial^2 u}{\partial t^2} = K \bullet \left(\frac{1}{N}\right)^2 \frac{\partial^2 u}{\partial z^2} + c\left(\frac{1}{N}\right)^2 \frac{\partial^3 u}{\partial z^2 \ell t}$$

as an approximation for the motion of the system. Rewritten in the form $$\rho\,\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial z}\,s(s,t)$$

(17) $$s(z,t) \triangleq K\,\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial t \partial z}$$

the equation describes the longitudinal motion of a visco-elastic bar; the term $s(z,t)$ represents the stress in the bar, here proportional to a linear combination of the strain and stress rate (Davis and Barry 1977) (Greenberg, MacCamy Mizel and 1968). The natural boundary conditions for (17) are in terms of $s(z,t)$ at $z=0$ and 1, i.e.,

(18) $s(0,t) = f_0(t),\ s(1,t) = f_1(t)$

the applied forces and $u(z,t)$, $u_t(z,t)$ at $t=0$.

Rescaling time at $t' = t/(k/c)$, the system (16) may be rewritten in dimensionless form as

(19) $$\frac{\partial^2 u}{\partial t^2} = a \frac{\partial}{\partial z} \frac{\partial}{\partial z} + \left(\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial z \partial t} \right) \quad 0<x<1,\ t<0$$

with $a = c^2/kMN^2$. And this may be written in matrix form as

(20a) $$\frac{\partial}{\partial t}\begin{bmatrix} u_z \\ u_t \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ a\frac{\partial}{\partial z} & a\frac{\partial^2}{\partial z^2} \end{bmatrix} \begin{bmatrix} u_z \\ u_t \end{bmatrix} \quad 0\le x\le 1,\ 0\le t$$

(20b) $$N_a[u_z + u_{zt}]\,(0,t) = f_0\,(t)$$
$$N_a\,[u_z + u_{zt}]\,(1,t) = f_1(t)$$
$$u_t\,(z,0),\ u(z,0) \text{ given}$$

Let A be the matrix differential operator on the right in (20a). Defining $H = L^2[0,1] \times L^2[0,1]$ with the energy inner product,

(21) $\quad < [{}^{u}_{v}] \; , \; [{}^{p}_{q}] > = \int_0^1 (a u \bar{p} + v \bar{q}) \, dz$

then A on H has domain

(22) $$D(A) = \{[{}^{u}_{v}]: u,v,v_x \text{ absolutely continuous } u_x, v_{xx} \in L^2 (0,1], \quad u + v_x = 0 \text{ at } x = 0,1\}$$

dense in H, A is a dissipative operator, and A is the infinitesimal generator of a class C_0 contraction semigroup T_t on H (Davis and Barry 1978, Theorem 1). Moreover, the resolvent

(23) $\quad R(s,A) = \int_0^\infty e^{-st} T_t \, dt$

of A may be computed explicitly. (See section 3 or Davis and Barry 1978.)

It is not possible to write the solution of (2) in the strong form

(24) $\quad$ "$\frac{\partial U}{\partial t} = AU + Bf(t)$"

where

(25) $$U = [u_z, u_t]^T , \; f = [f_0, f_1]^T$$
$$B: R^2 \to H$$

since the boundary forces correspond to generalized function "inputs". Using L^{-1} as the inverse Laplace transform, Davis and Barry treat (20) in the form

(26) $\quad U(t) = T_t \, U(0) + L^{-1} \{G(s;z) \, \hat{f} \, (z)\}$

where G(s;z) represents the "transfer function matrix" associated with the boundary value problem (20).

The (optimal) control problem involves minimizing variations in the stress distribution throughout the system. The quantity

$$(27)\quad \int_0^1 \alpha^2 s^2 \;(z,t)\; dz = \int_0^1 \alpha^2 (u_z + u_{zt})^2\, dz \qquad \alpha = a/n$$

corresponds to the total stress in the system. Recalling the original approximation (13)-(16), we have the correspondence

$$(28)\quad \int_0^1 \alpha^2 (u_z + u_{zt})^2 dz \sim N \sum_{i=1}^{N} [\tfrac{K}{M}\,(x_i - x_{i-1}) + \tfrac{C}{M}\,(\dot{x}_i - \dot{x}_{i-1})]^2$$

and so, the natural quadratic cost functional is

$$(29)\quad J = \int_0^\infty \{\, |f_0(t)|^2 + |f_1(t)|^2 + \frac{1}{N}\int_0^1 \alpha^2\,(u_z + u_{tz})^2\, dz\}dt$$

This formulation includes stress contributions from spatial modes of all wavelengths. In most physical systems high order modes will have a negligible contribution to the overall behavior. Using π_p to denote projection onto the subspace of H spanned by the first p eigenfunctions of A, and defining the system output as

$$(30)\quad y\,(z,t) = \alpha[1,\partial_z]\;[\pi_p U]\;(z,t)$$

the final formulation used by Davis and Barry (1978) is

$$\min_{f(\cdot)} \int_0^\infty \{\, |f_0(t)|^2 + |f_1(t)|^2 + \frac{1}{n}\int_0^1 |y(z,t)|^2 dx\}\; dt$$

$$U(z,t) = T_t\, U(0)(z) + L^{-1}\{G(s;z)\,\hat{f}\,(z)\}\,(t)$$

$$y(z,t) = \alpha[1,\partial_z]\,[\pi_p U]\,(z,t)$$

$$U(0) \in D(A) \subset L^2[0,1] \times L^2[0,1]$$

The resulting optimization problem is a distributed control problem with state cost restricted to a finite dimensional subspace.

Davis and Barry (1978) compute the optimal control law for this problem using the spectral factorization algorithm described earlier. A key step in the procedure is application of a numerical algorithm for spectral factorization due to F. Stenger. In the next subsection we summarize Stenger's algorithm.

2.2 Stenger's Algorithm for Spectral Factorization

To evaluate the control law for a given problem modeled as in the last subsection, we must compute the spectral factor F(s) appearing in equation (10). That is,

$$(32) \qquad F^*(s)\,F(s) = H(s) = I + G^*(s)\,G(s)$$

where G(s) is the transfer function of the system being controlled. The first problem is to determine conditions under which the spectral factor exists. Since G(s) is the transform of a real vector valued function, which we assume to be integrable and square integrable, it follows that $S(s) = G^*(s)G(s)$ is a Hermitian positive semidefinite (matrix valued) function and G^*G is the transform of a function which is in $L^1 \cap L^2$.

Since $S(s)$ is the transform of a function in L^2, it follows from the classical theory of Gohberg and Krein (1960) that $H(s)$ has a unique spectral factorization of the form (32) where

$$F^{\pm 1} - I \in F(L_1^+) \quad (= \text{Fourier transforms of } L_1^+ \text{ functions})$$
(33)
$$F(i\omega) = F(-i\omega)$$

where L^+ denotes those functions in L with positive support. As noted in (Davis and Dickenson 1983), the assumptions on $S(s)$ in fact imply

(34) $$F^{\pm 1} - I \in F(L_1^+ \cap L_2^+)$$

and $F(i\omega) = F(-i\omega)$. These conditions, therefore, settle the question of existence and uniqueness of the spectral factor.

In (Davis and Dickenson 1983) an iterative algorithm was given for computing the spectral factor. Since this method is at the heart of our computational programs, and since it makes use of Stenger's algorithm, we shall develop it here. The starting point for the iteration in (Davis and Dickenson 1983) is the Newton-Raphson iteration for the solution of the algebraic Riccati equation (4); that is,

(35) $$K_{n+1}(A - BB^*K_n) + (A - BB^*K_n)^* K_{n+1} = -C^*C - K_n BB^* K_n$$

From this a simple calculation leads to the desired form of the iteration for the spectral factor (see Davis and Dickenson 1983, pp. 290-291)

(36) $$F_{n+1} = P_+[(F_n^*)^{-1} S(F_n)^{-1}] F_n$$

where $P_+[\cdot]$ is the causal projection operator defined on the convolution algebra $I \oplus L^1$, or on L^2, by

$$(37)\quad P_+ \left[I + \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt\right] = I + \int_0^{\infty} f(t)\, e^{-i\omega t} dt$$

Stenger's algorithm is used to provide a numerical approximation to the causal projection operator. Before discussing that, we note that under the assumptions on $G(s)$, and therefore on $S(s)$, that the iteration may be shown to converge from a suitable initial guess to the uniquely defined (in L^2) causal spectral factor $F(s)$.

The algorithm (36) has a particularly simple form. As noted in (Davis and Dickenson 1983), the computation of $P_+[\cdot]$ is the most difficult step, but Stenger's result takes care of that. The numerical approximation in (Stenger 1972) takes the form

$$(38)\quad P_+ [f]\ (\omega) = f_t\ (\omega) \sim \sum_k \hat{f}(kh + \tfrac{1}{2}h) \sum_m \frac{r_m}{(\omega - kh - \alpha_m h)}$$

Here h is the step size and r_m and α_m are parameters defined by Stenger. Specifically,

$$(39)\quad \alpha_m = \frac{1}{1 - iq^m} \qquad r_m = -\frac{\pi h}{4k^{\frac{1}{2}}K} \quad q^{m} \alpha_m^2$$

where q is a parameter chosen in the algorithm and

$$(40)\qquad K = (a/b)^2 \qquad\qquad k = \tfrac{1}{2}\pi b^2$$

$$a = 2 \sum_{m=1}^{\infty} q^{(m-1)} \qquad\qquad b = 1 + 2 \sum_{m=1}^{\infty} q^{m^2}$$

The step size is chosen based on the bandwidth of the transfer function $\hat{f}(s)$. If, in (38), one chooses to sample the projection $\hat{f}_+$ at the same sample points as $\hat{f}$, then, as observed in (Davis and Dickenson 1983)

$$(41) \quad \hat{f}_+(kh + \tfrac{1}{2}h) \sim \sum_j \hat{f}(jh + \tfrac{1}{2}h) \sum_m \frac{r_m}{(k-j)h-\alpha_m h + \tfrac{1}{2}h}$$

and it is clear that the required calculation is a convolution. Since the range of sample points is finite this is naturally implemented by a fast Fourier Transform (FFT); and this substantially improves the computational time.

Since we must compute (F_n^*) $S(F_n)^{-1}$ in the iteration (36) for the spectral factor, it is best to rewrite the iteration as

$$(42) \quad (F_{n+1})^{-1} = (F_n)^{-1} \; (I + P_+[(F_n^*)^{-1} \; S(F_n)^{-1} - I])^{-1}$$

and execute it in this form. As observed in (Davis and Dickenson 1983), the last factor in this expression is a perturbation of the identity (since $(F_n^*)^{-1}S(F_n)^{-1} - I \to 0$), and this has natural advantages in the numerical realization of the iteration.

As suggested in (Davis and Dickenson 1983), a suitable choice for the initial guess for the spectral factor F_0 is the diagonal matrix whose elements are (scalar) spectral factors of the corresponding diagonal elements of S. This choice implies that the matrix $(F_n^*)^{-1}S(F_n)^{-1}$ is a matrix with ones on the diagonal and with all the off-diagonal elements less than one in magnitude. This tends to prevent the iteration from blowing up. The diagonal factors may be obtained by an FFT implementation of the scalar algorithm in (Stenger 1972).

The method as described here was implemented directly on the problem of controlling the dynamics of the Euler-Bernoulli beam. The results are shown in the next section. A careful consideration of Stenger's method suggests an alternative implementation of the algorithm which takes advantage of the occurence of Hilbert transforms in the course of the computations and the effective use of these transforms in the representation of the causal spectral factor. This observation permits an efficient numerical realization of the spectral factorization procedure. We shall develop this in the context of design of stabilizing controls for a two dimensional flexible structure. This result and the associated algorithm are reported in section 4.

3. Control of a One-Dimensional Structure

The control of simple one dimensional structures serves to illustrate the general analytical methods in the simplest form. One dimensional models can also represent certain components, e.g., flexible beams, which appear in composite large space structures; and they may represent certain two or three dimensional structures with natural symmetry. In this section we consider a simple system, the Euler-Bernoulli beam in detail, working through the computation and simulation of the optimal stabilizing feedback gain.

3.1 Control of a Flexible Beam

The dynamics of an undamped flexible beam undergoing transverse notion are described by the Euler-Bernoulli partial differential equation

$$\begin{aligned} &m\, u_{tt}(t,x) + EIu_{xxxx}(t,x) = f(t,x) \\ &0 \leq x \leq L,\ t \geq 0 \end{aligned} \tag{1}$$

where $u(t,x)$ is the transverse displacement of the beam, $f(t,x)$ is an applied force distribution, m is the mass per unit length, I the moment of inertia, E the modulus of elasticity, and L is the beam length. To facilitate comparison of our results with earlier work (Balas 1978b), we shall assume that m, E, I, and L are all unity. The boundary conditions for pinned support are

$$\begin{aligned} &u(t,0) = 0 = u(t,1) \\ &u_{xx}(t,0) = 0 = u_{xx}(t,1) \end{aligned} \tag{2}$$

The beam is controlled by a single point actuator

(3) $f(t,x) = \frac{1}{\sqrt{2}} \delta(x-a)v(t)$, $0<a<1$

and a single sensor measures displacement

(4) $y(t) = u(t,b)$, $0<b<1$

The system is deterministic and actuator and sensor dynamics are not modeled.

Balas (1978b) designed a feedback controller for the first three modes of the beam which minimized the (unweighted) energy in those modes. His controller includes a six-dimensional Luenberger observer to reconstruct the state. The energy in the fourth (residual) mode increases rapidly due to spillover effects.

Our approach to this problem is based on the infinite dimensional model. Taking the Laplace transform of (1), we have

$$
\begin{aligned}
&U_{xxxx}(s,x) + s^2 U(s,x) = F(s,x) \\
(5) \quad &U(s,0) = 0 = U(s,1) \\
&U_{xxx}(s,0) = 0 = U_{xx}(s,1)
\end{aligned}
$$

the Green's function for (1) (5) satisfies

(6) $G_{xxxx}(s,x|x') + s^2 G(s,x|x') = \delta(x-x')$

Consider G in the form

(7) $G(s,x|x') = \begin{cases} A \sinh \sqrt{s}\, x) & 0 \le x \le x' \\ B \sinh \sqrt{s}\, (1-x)) & x' \le x \le 1 \end{cases}$

with A and B complex constants to be determined. Note that the boundary conditions in (5) are satisfied by (7). At $x=x'$ we have

$$(8) \qquad \int_{x'-\varepsilon}^{x'+\varepsilon} G_{xxxx}\, dx + s^2 \int_{x'-\varepsilon}^{x'+\varepsilon} G\, dx = 1$$

(See Tai (1971) . From (6) (8) we have

$$(9) \qquad G_{xxx} \Big|_{x'-\varepsilon}^{x'+\varepsilon} = 1$$

and from this

$$(10) \begin{pmatrix} A \\ B \end{pmatrix} = \frac{-1}{s^{3/2} \sinh s} \begin{pmatrix} \sinh\ s\ (1-x') \\ \sinh\ s\ x' \end{pmatrix}$$

which gives the Green's function.

The transfer function relating the input $f(t,x)$ in (3) to the output $y(t)$ in (4) may be written down immediately from $G(s,x|x')$.

$$(11) \quad Y(s) = U(s,b) = \int_0^1 C(s,x|x')\, F(s,x')dx' = \frac{1}{\sqrt{2}} G(s,b|a)\ V(s)$$

with $V(s)$ the Laplace transform of $v(t)$. Identifying

$$(12) \quad T_{ab}\ (s) = G(s,b|a)$$

we have

$$(13)\quad T_{ab}\ (s) = \begin{cases} \dfrac{-(\sinh a\ \ s)\ \sinh\ (1-b)\ s)}{\sqrt{2}\ s^{3/2}\ \sinh\sqrt{s}}, & x'=a<b=x \\[2ex] \dfrac{-(\sinh\ (1-a)\ \ s)\ \ (\sinh b\ \ s)}{\sqrt{2}\ s^{3/2}\ \sinh\sqrt{s}}, & x=a>b=x \end{cases}$$

Balas (1978) uses $a = \frac{1}{6}$, $b = \frac{5}{6}$; and for this case we have

$$(14)\quad T_B(s) = T_{\frac{1}{6}\,\frac{1}{5}}\ (s) = \frac{-(\sinh\sqrt{s/6})^2}{2\ s^{3/2}\ \sinh\sqrt{s}}$$

To use the Davis-Stenger algorithm as described in section 2, we must compute the spectral factorization of

$$(15)\quad H(s) = 1 + T_B^*\ (s)\ T_B\ (s) \triangleq F^*\ (s)F(s)$$

Substituting, we have

$$(16)\quad H(i\omega) = 1 + \tfrac{1}{2}\ \frac{(\sinh\sqrt{i\omega/6})^2\ \ (\sinh\sqrt{-i\omega/6})^2}{(i\omega)3/2\ \ (-i\omega)3/2\ \sinh\sqrt{i\omega}\ \sinh\sqrt{-i\omega}}$$

and the computational problem reduces to computing the spectral factor $F^*(s)$ from (16), and then computing the optimal stabilizing feedback gain

$$(17)\quad B^*K = \frac{1}{2\pi}\int_{-\infty}^{\infty}[F^*\ (i\omega)]\ \ G^*\ (i\omega)\ \ CR\,(i\omega,\ A)\ \ d\omega$$

by numerical integration.

3.2 Numerical Results

In the computations and simulations which follow, we consider the same beam parameters used by Balas (unit length, with parameters normalized to one, and zero damping). The numerical requirements of the algorithm are not changed for more realistic choices of parameters. In addition to the case considered by Balas (with one controller and one observer at opposite ends of the beam), we also consider the effect of increasing the number of controllers and observers, and finally the effect of delays in the control loops.*

Practical implementation of the algorithm requires frequency sampling of the transfer function and spectral factors, and spatial and frequency sampling of the resolvent. It was determined experimentally that for the given beam parameters, a frequency range of plus/minus 30HZ is adequate, since the gain from input to output in the range beyond this is insignificant, see Figure 3.1. The figure also indicates a very smooth dependence of the transfer function on frequency, so that the 256 sample points used in the program are quite adequate. Spatial discretization is done using 100 equidistant points.

An example of the feedback gain is given in Figure 3.2, for the velocity state variable, and in Figure 3.3 for the displacement state variable. (See Appendix A for other gains corresponding to different

*The issue of the effects of the delay on the control action and the system stability was raised by Dr. J. Burns, formerly with AFOSR, and now with VPI&SU, Blacksburg, VA. We are grateful to Professor Burns for his inputs on this problem.

arrangements of the controllers and outputs.) A sharp peak at the point of observation for the deflection state indicates the control effort to reduce the deviation at this point to zero, since only this point contributes to the optimization cost. In this case we have penalized the state deviation at the observation points (in the cost criterion) 500 times more than the control, so this is "cheap control."

The feedback gains, one for the speed state and the other for the deflection state are integral operators as defined by (17). These gains are computed off-line. Computation of the input function for a given time requires evaluation of the integral operator B K acting on the state. This is accomplished in the program by approximating the integral as a sum of piecewise constant functions.

In controlling a physical beam one would need an observer for the deflection and velocity variables that would use (point) observations of the beam deflection as inputs. For the simulation results here the deflection and its derivative are obtained by numerical integration.

The case studied by Balas, where the controller and the observer are at the opposite ends of the beam, exhibits poor observability and controllability, which is reflected in the large control efforts and long settling times to stabilize the motions (see Figure 3.6). Using more observers and controllers substantially improves the "controllability" of the system. For example, in the case of three collocated, equidistant controllers/measurements, the margin of improvement can be seen by comparing Figure 3.5 with Figure 3.6. See also Appendix A for the time responses corresponding to these two cases.

Delays in physical control loops are inevitable due to the finite time necessary to process measurements and compute the resulting controls. Analytical treatment of delays using time domain models is not nearly as convenient as it is in the frequency method described here. For example, if the delay is T, then we need only multiply each element of matrix H(s) in (15) by a factor exp(-sT). Therefore, H(s) is invariant with respect to the delay, and the critical part of the gain computation algorithm, i.e., spectral factorization, need not be recomputed for the delayed case. Of course, the delay appears in the transfer function, and so, a new gain must be computed for each different delay. (Numerical results displaying effects of delays are discussed later.)

To validate the program, we simulated the response of a beam subject to an initial disturbance in the form of one of the spatial modes, and with a feedback based on the optimal control discussed above. Figure 3.5 is an example of a time variation of the beam at the observation point. (Other such examples are given in Appendix A.) Note that without the control this response would represent an undamped oscillatory motion since the beam model does not include damping. Damping has a stabilizing effect in this system; and the control action is enhanced if damping is present.

The plots summarizing numerical work on the example problem appear in several groups in Appendix A, and they are grouped as follows. For a given delay, and a given number and position of controls and observations, we first display gains for deflection and velocity states, both for each of the inputs. Next is a group of plots showing the beam response to the optimal control when the beam is initially displaced in the form of one of the first three spatial modes. (Significant components of the matrix H(i) are well below 30Hz, i.e., below the frequencies induced by the 4th spatial mode.) The meaning of the plot titles is as follows: "Beam at x, yth mode," means that the beam is initially displaced by the y-th spatial mode, and the deflection of the beam is observed as a function of time at point x. On the control plot we indicate the position of the point control and the time evolution of the control at that point.

While the main purpose of conducting these experiments was to verify the control algorithm, several phenomena were observed from these experiments. Comparing responses of higher order modes with those of the lower order modes, it is evident that more energy is needed to control higher order modes (note that our model has zero damping). This reflects the poor controllability and observability of the higher order modes. Second, the gains for the deflection state have pronounced peaks at the observation points, suggesting use of localized - decentralized feedback. Unfortunately, the speed gains indicate much more spatial coupling, suggesting that decentralized control schemes may not be effective (at least for the parameter ranges used in this problem). Third, the delay has a substantial effect on the performance of the control. (Compare any case with delay from Appendix A with a case with no delay.) Nevertheless, the stability margin is remarkably wide. An example from Appendix A indicates that a delay equal to one half period of the highest mode in the chosen spectrum does not destabilize the system.

The numerical results presented above affirm the Davis-Stenger algorithm as a practical tool for vibration control of flexible structures, represented here by an Euler beam model. The results of this algorithm provide a means for assessing effects of controller/observer placement on the system performance, as well as give stabilizing feedback gains, once the controller locations have been selected. It was also demonstrated that the underlying model allows an efficient treatment of delays in the control loop.

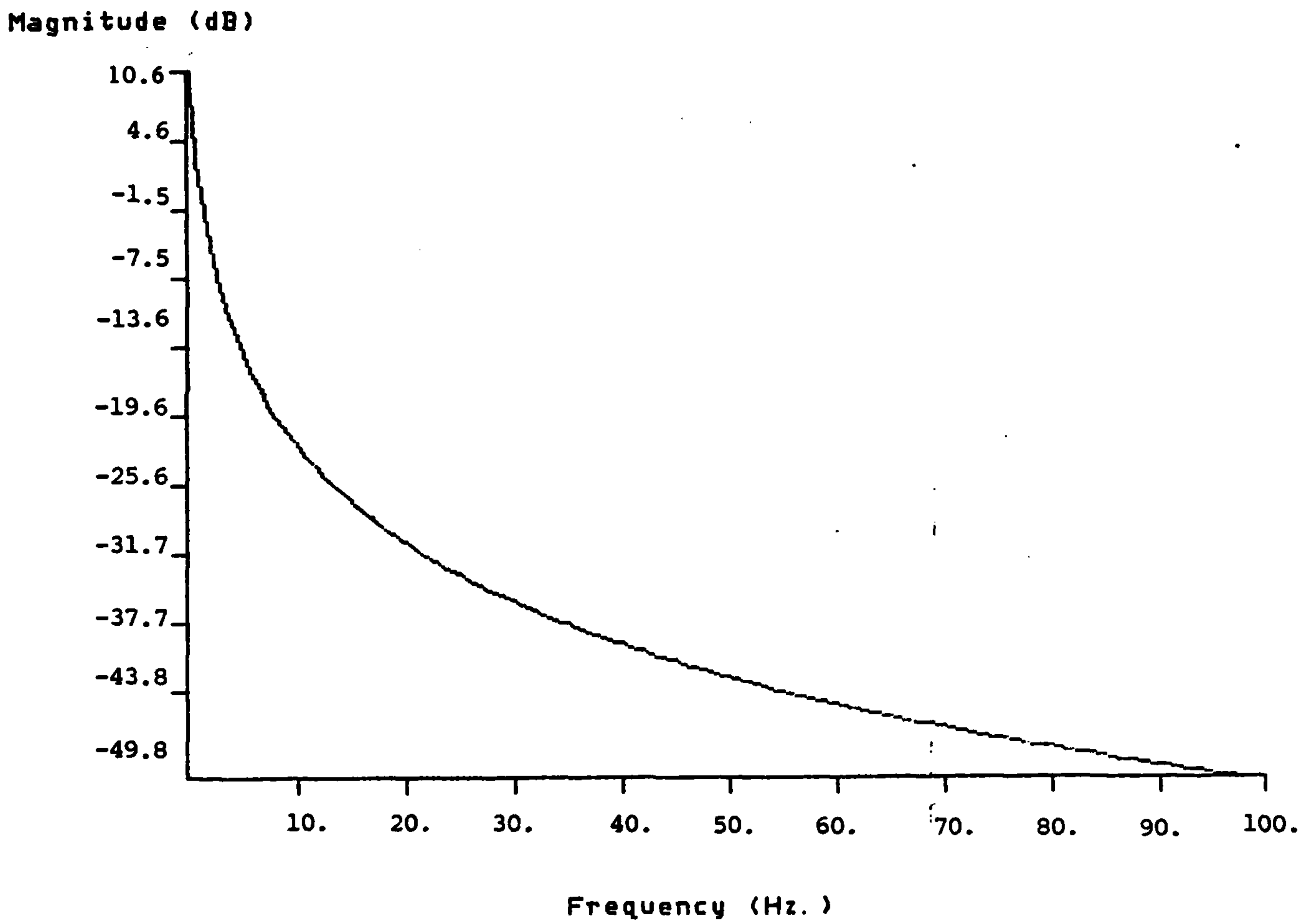


Figure 3.1 $|H(j\omega)|$ versus ω for the case of one controller at .8 and output at .2

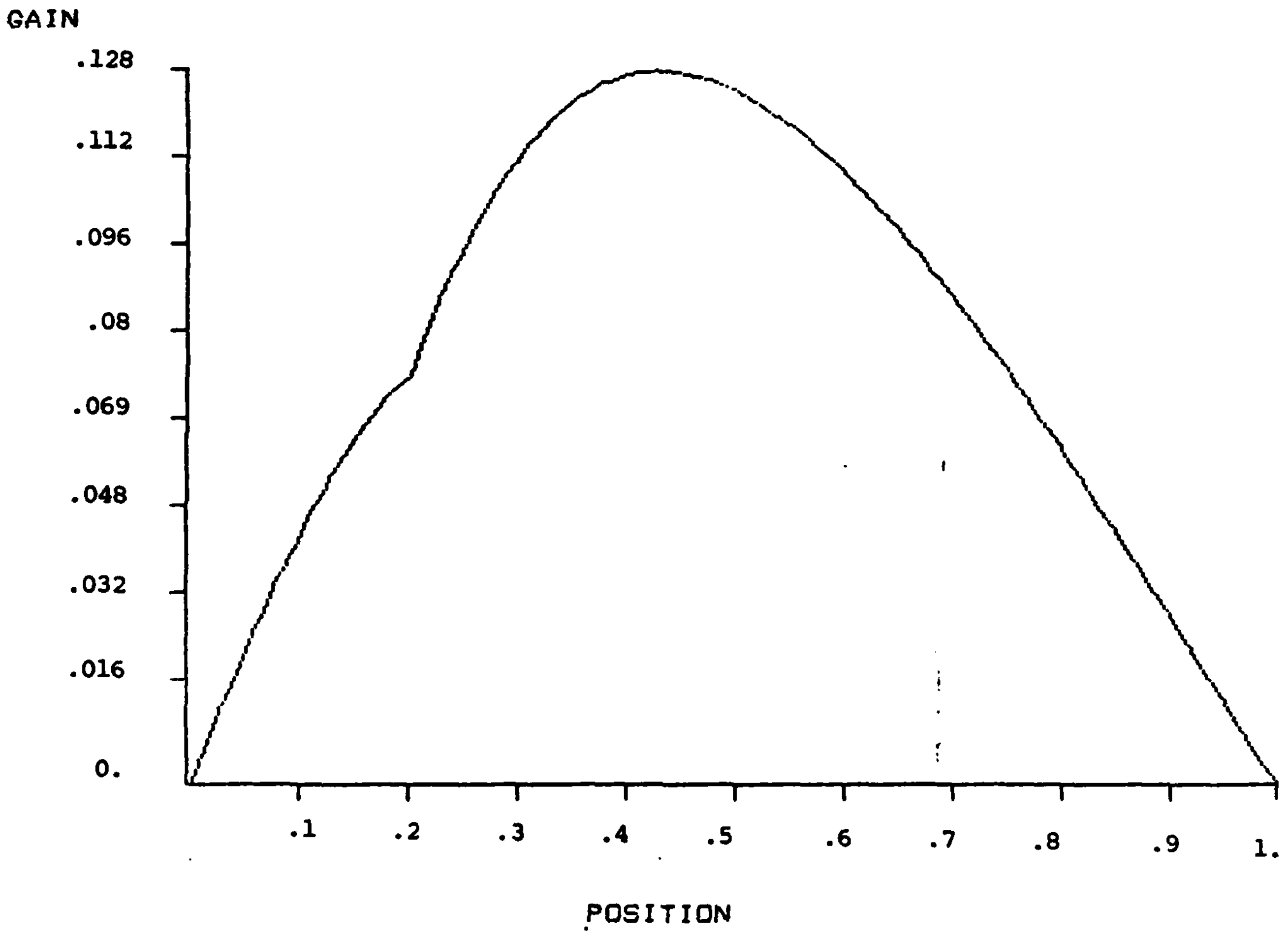


Figure 3.2. Gain profile for the velocity state; Actuator at .8, output at .2

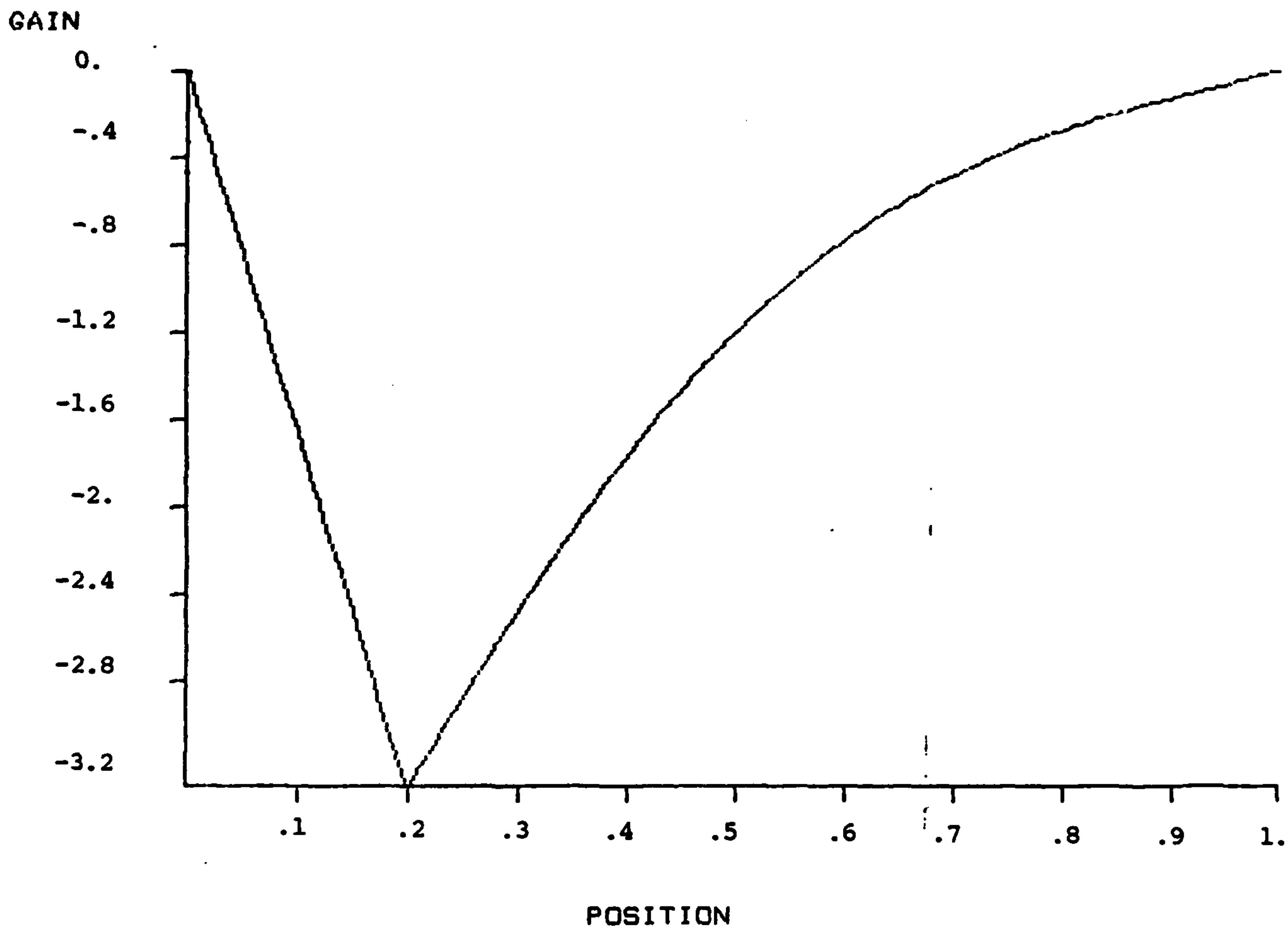


Figure 3.3. Gain profile for the deflection state; One Actuator located at .8, output located at .2.

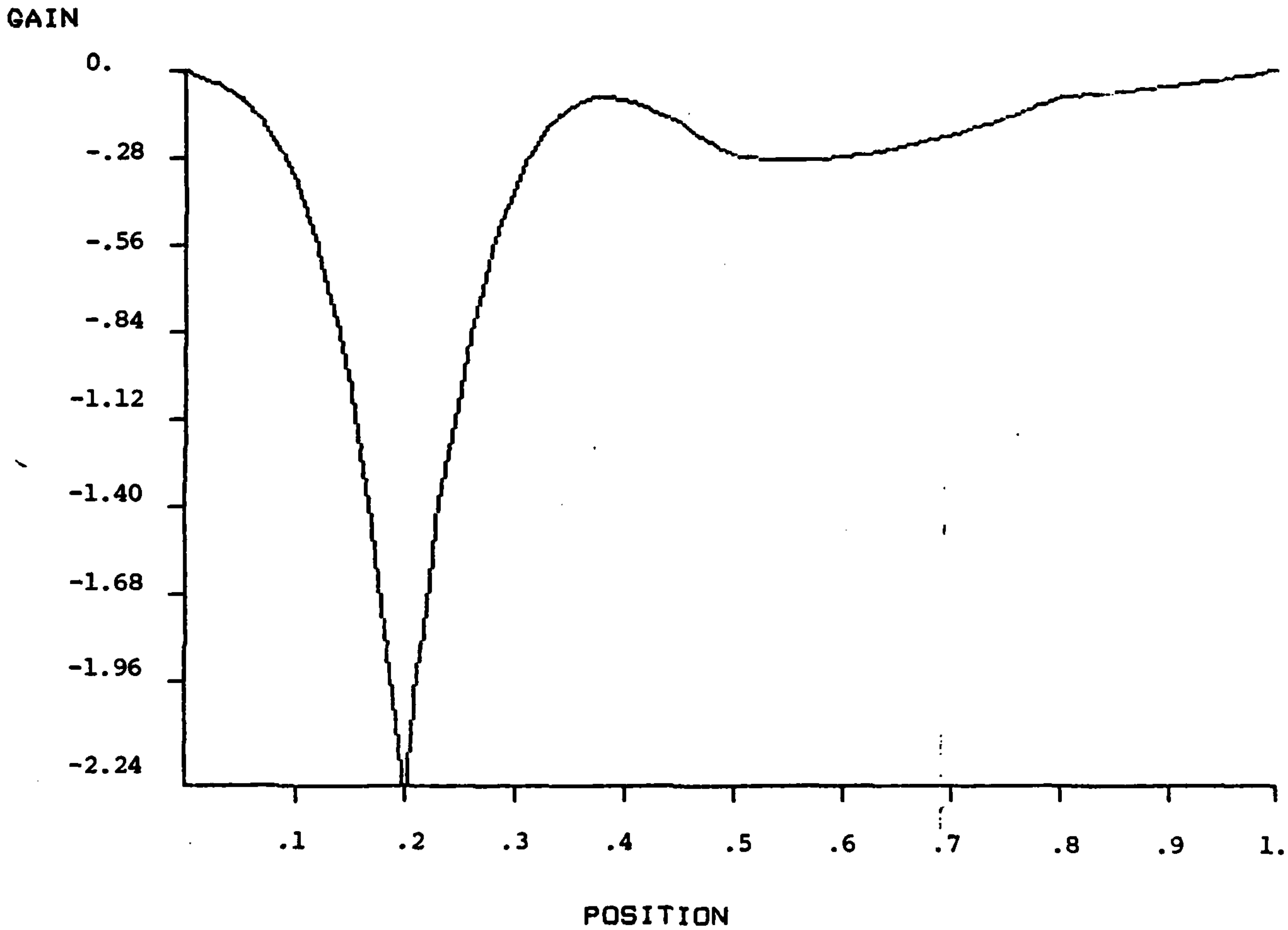


Figure 3.4. Gain profile for the deflection state and first output; three collocated actuators/outputs.

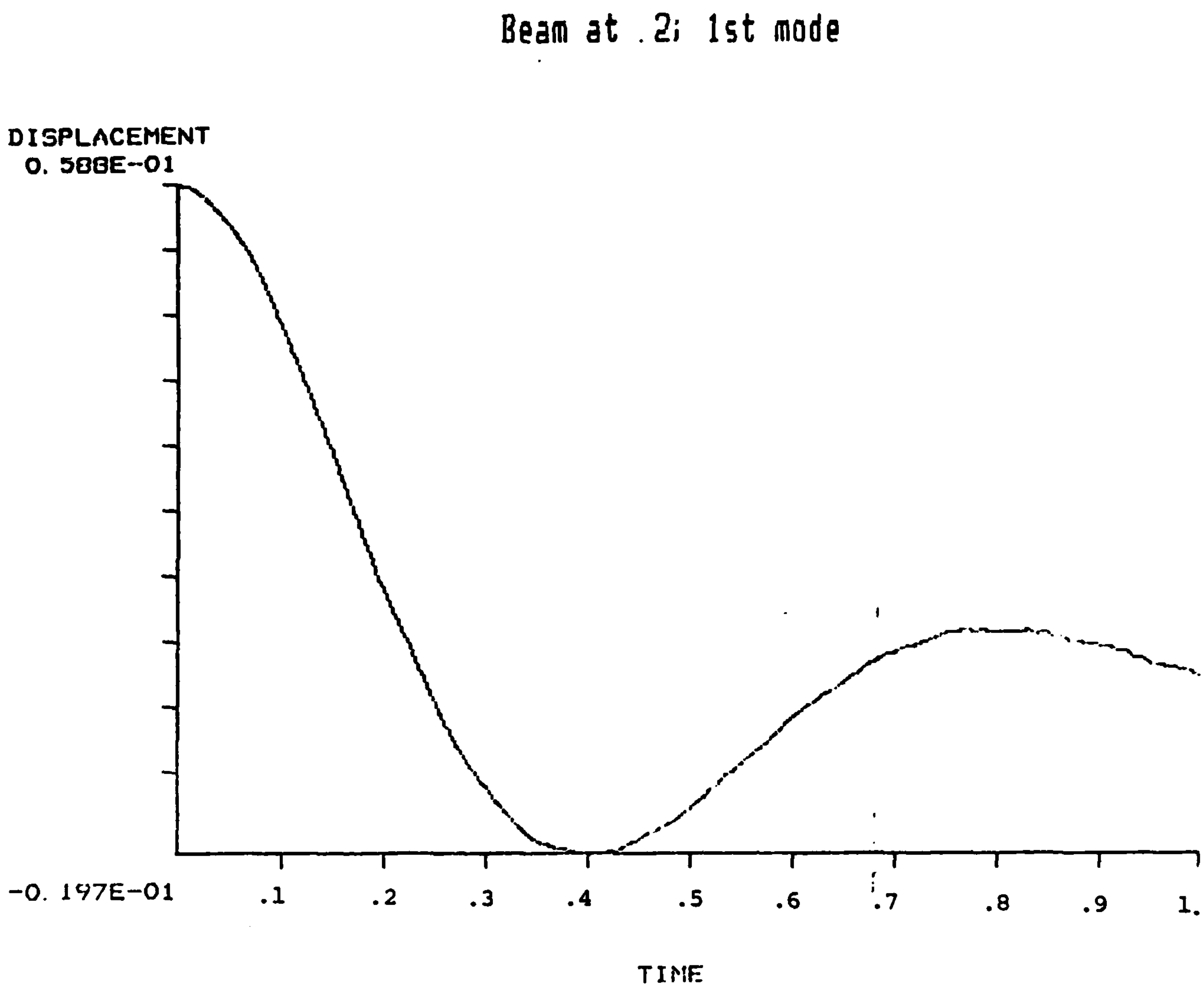


Figure 3.5. Time response of the beam; three actuators.

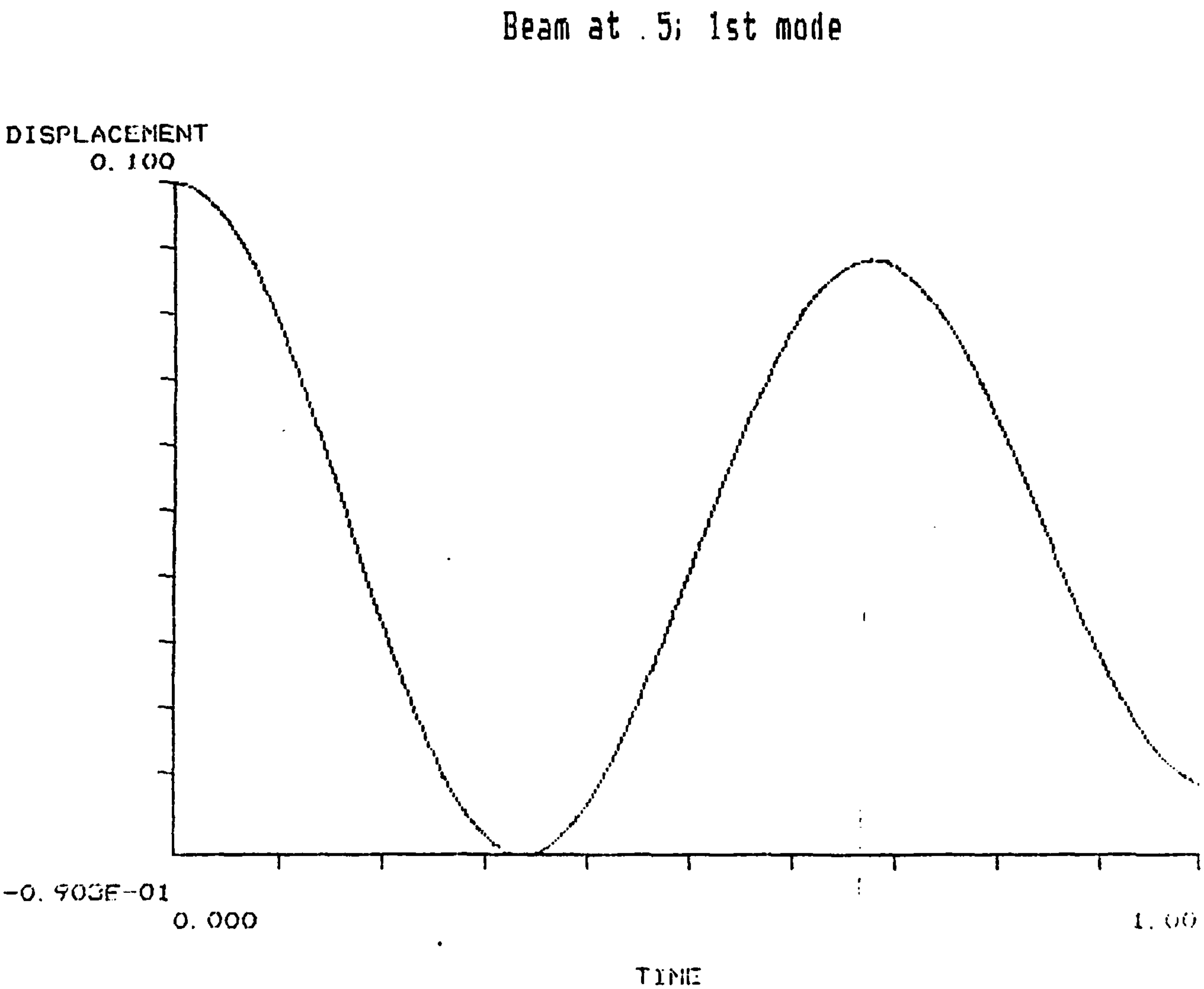


Figure 3.6. Time response of the beam; one actuator.

4. Control of a Two-Dimensional Prototype System

In this section we adapt our frequency domain control system design procedure to treat a prototype two dimensional flexible system - a membrane/mesh whose dynamical behavior is sensed and controlled by electrostatic forces. The model is patterned after an experimental system studied by Lang and Staelin (1982b) as a paradigm for an electrostatically controlled large aperature reflecting satellite antenna.

While the starting point for the control system computations is similar to that for the Euler beam, our analysis takes a somewhat different tack. We shall exploit the appearance of Hilbert transforms in the derivation of the spectral factor and the simple way in which these transforms can be used to represent the spectral factor appearing in the expression for the feedback gain. As we shall see, there are some significant computational advantages obtained in this modification of the Davis-Stenger procedure.

In the next subsection we describe the model and compute the transfer function. In the following subsection we write the solution for the mesh dynamical system in terms of the eigenfunctions of the evolution operator. This provides an effective and accurate basis for numerically simulating the (controlled) system dynamics. It is superior to numerical solution of the PDE model by finite difference methods.

4.1 Dynamics of a Vibrating Membrane

The physical structure of the prototype experimental system used by Lang and Staelin (1982a,b) was described in section 1.3. We shall describe the mathematical analysis of the system here. The linearized equation describing the dynamics of the voltage controlled mesh is

$$(1)\quad \frac{\partial^2 h}{\partial t^2} = \frac{T_\alpha}{M}\frac{\partial^2 h}{\partial \alpha^2} + \frac{T_\beta}{M}\frac{\partial^2 h}{\partial \beta^2} - \frac{W}{M}\frac{\partial h}{\partial t} + \frac{2\varepsilon_0 V}{M\tilde{H}^3}\, h - \frac{\varepsilon_0 V}{M\tilde{H}^2}\quad v$$

$h=0$ on boundary of $[0,L] \times [0,\hat{L}]$

$h(0,\alpha,\beta) = 0,\ h_t(0,\alpha,\beta) = 0$

where the boundary conditions reflect the fact that the mesh is pinned along its boundary. To reduce the computations, we make a change of coordinates to remove the derivative term in (1); that is,

$$(2)\quad h(t) = f(t)\ \exp\left\{\frac{-W}{2M}\,t\right\}$$

We obtain $$\frac{\partial^2 f}{\partial t^2} = \frac{T_\alpha}{M}\frac{\partial^2 f}{\partial \alpha^2} + \frac{T_\beta}{M}\frac{\partial^2 f}{\partial \beta^2} + \frac{2\varepsilon_0 V^2}{M\tilde{H}^3}\, f - \frac{\varepsilon_0 V}{M\tilde{H}^2}\ .\exp\left\{-\frac{W}{2m}\,t\right\}\ v$$

(3) $f=0$ on boundary of $[0,L] \times [0,\hat{L}]$

$f(0,\alpha,\beta) = 0\ \ f_t(0,\alpha,\beta) = 0$

Let us define the parameters

(4) $a_\alpha = T_\alpha/M,\ a_\beta = T_\beta/M,\ \gamma^2 = 2\varepsilon_0 V^2/M\tilde{H}^3$

$$-a_\beta u = \frac{\varepsilon_0 V}{M\tilde{H}^2}\ \exp\frac{-W}{2M}\,t \quad v$$

and using these in (3) we obtain

$$
(5)\quad
\begin{aligned}
&\frac{\partial^2 f}{\partial t^2} = a_\beta \frac{\partial^2 f}{\partial t^2} + a_\alpha \frac{\partial^2 f}{\partial \alpha^2} + \gamma^2 F + a_\beta u \\
&f = 0 \text{ on boundary of } [0,L] \times [0,\hat{L}] \\
&f = f_t = 0 \text{ at } t = 0
\end{aligned}
$$

It is possible to reduce the equation further to the standard form for wave equations; but we shall not do this since it will complicate the boundary conditions.

Taking the Fourier transform in (5), we obtain

$$
(6)\quad
\begin{aligned}
&(-i\omega)^2 F(\omega) = a_\beta \frac{\partial^2 F}{\partial \beta^2} + a_\alpha \frac{\partial^2 F}{\partial \alpha^2} + \gamma^2 F + a_\beta u \\
&F = 0 \text{ on boundary of } [0,L] \times [0,L]
\end{aligned}
$$

By splitting F and U into real and imaginary parts, respectively, we are led to the following equation

$$
(7)\quad a_\alpha \frac{\partial^2 H}{\partial \alpha^2} + a_\beta \frac{\partial^2 H}{\partial \beta^2} + (\gamma^2 + \omega^2) H = -a_\beta^2 v
$$

Our immediate objective is to find the Green's function corresponding to the boundary value problem (7).

To accomplish this, we shall reduce the problem to a one dimensional Sturm-Liouville problem. Consider the operator

$$
(8)\quad
\begin{aligned}
&L_\alpha u = -a_\alpha \frac{\partial^2 u}{\partial \alpha^2} - (\gamma^2 + \omega^2) u \\
&\text{with } u(0) = u(\ell) = 0
\end{aligned}
$$

Using this in (7), we have

$$
(9)\quad
\begin{aligned}
&a_\beta \frac{\partial^2 H}{\partial \beta^2} - L_\alpha H = -a_\beta u \\
&H = 0 \text{ on the boundary}
\end{aligned}
$$

and so, the Green's function satisfies the PDE

$$(10)\quad \begin{cases} a_\beta \dfrac{\partial^2 K}{\partial\beta^2} - L_\alpha K = -\delta(\alpha-\xi)\,\delta(\beta-\eta) \\ K_{\alpha,\beta} = 0 \text{ on the boundary of } [0,\ell] \times [0,\hat{\ell}] \end{cases}$$

Let $\phi_k(\alpha)$ be the eigenfunctions of the operator L_α, $k = 1,2,\ldots$.

We shall compute the Green's function in the form

$$(11)\quad K(\alpha,\beta,\xi,\eta) = \sum_{k=1}^{\infty} a_k(\beta)\,\phi_k(\alpha)$$

We define the weighted inner product

$$(12)\quad \langle \phi,\psi \rangle = \iint_\Omega a_\beta\,\phi\,\psi \quad \text{where } \Omega = [0,\ell] \times [0,\hat{\ell}]$$

Substituting the expression (11) for K into (10), we obtain

$$(13)\quad a_\beta \sum_{k=1}^{\infty} a''_k(\beta)\,\phi_k(\alpha) - \sum_{k=1}^{\infty} a_k(\beta)\,\phi_k(\alpha) = -\delta(\alpha-\xi)\,\delta(\beta-\eta)$$

But $\phi_k(\alpha)$ are eigenfunctions of L

$$(14)\quad L\,\phi_k(\alpha) = \lambda_k\,a_\beta\phi_k(\alpha)$$

Multiplying both sides of (13) by $\phi_j(\alpha)$, $j \neq k$ and integrating over α we obtain

$$(15)\quad \begin{aligned} &\int_0^L [a_\beta \left(\sum_{k=1}^{\infty} a''_k(\beta)\,\phi_k(\alpha) \right) \phi_j(\alpha) - \int_0^\ell \lambda_k\,a_\beta\,a_k(\beta)\,\phi_k(\alpha)\,\phi_j(\alpha)]\,d\alpha \\ &= -\int_0^L \delta(\alpha-\xi)\,\delta(\beta-\eta)\,d\alpha \end{aligned}$$

which implies (using the orthogonality of ϕ and the properties of the delta function)

$$
(16) \quad \begin{aligned} & a_k''(\beta) - \lambda_k a_k(\beta) = -\phi_k(\xi)\, \delta(\beta - \eta) \\ & a_k(0) = a_k(\hat{L}) = 0 \end{aligned}
$$

This is a classical Sturm-Liouville problem.

The eigenfunctions of the operator L satisfy

$$
(17) \quad \begin{aligned} & L\phi_k = \lambda_k a_\beta \phi_k = -a_\alpha \phi_k'' - (\lambda^2 + \omega^2)\phi_k \\ & \phi_k(0) = \phi_k(\ell) = 0 \end{aligned}
$$

It is a simple calculation to show that the eignevalues are

$$
(18) \quad \lambda_n = \frac{a_\alpha}{a_\beta} \frac{n^2\pi^2}{\ell^2} - \frac{\gamma^2 + \omega^2}{a_\beta} \qquad n = 1,2,...
$$

and the eigenfunctions are

$$
(19) \quad \phi_n(\alpha) = \sqrt{\frac{2}{\ell}} \sin\left(\frac{n\pi\alpha}{\ell}\right) \qquad n = 1,2,...
$$

Now we have to solve the Sturm-Liouville problem (16). Referring to (18), we have to consider the three cases: $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. Note that for each fixed ω there is only a finite number of negative eigenvalues. To solve (16), it suffices to solve

$$
(20) \quad \begin{aligned} & a'' - \lambda a = - \delta(\beta - \eta) \\ & a(0) = a(\hat{\ell}) = 0 \end{aligned}
$$

We have dropped the indices for convenience and the term $\phi_n(\xi)$ which will be handled by multiplying the resulting Green's function by the same factor.

Case 1: $\lambda = \mu^2 > 0$

It is a simple matter to show that the Green's function associated with problem (20) is

$$(21) \quad G(\beta\eta) = \frac{1}{\mu \operatorname{sh}(\mu\ell)} \begin{cases} \operatorname{sh}(\mu(\ell-\eta)) \operatorname{sh}(\mu\beta) & 0<\beta<\eta \\ \operatorname{sh}(\mu\eta) \operatorname{sh}(\mu(\ell-\beta)) & \eta<\beta<\ell \end{cases}$$

where $\operatorname{sh}(x) = \sinh(x)$. To obtain the desired Green's function associated with (16), we simply multiply (21) by $\phi_n(\xi)$. This gives

$$(22) \quad G_n(\beta,\eta) = \frac{2 \sin(\frac{n\pi}{\ell})}{\ell \mu_n \operatorname{sh}(\mu_n \ell)} \begin{cases} \operatorname{sh}(\mu_n(\ell-\eta)) \operatorname{sh}(\mu_n\beta) & 0<\beta<\eta \\ \operatorname{sh}(\mu_n\eta) \operatorname{sh}(\mu_n(\ell-\beta)) & \eta<\beta<\ell \end{cases}$$

where $\mu_n = \sqrt{\lambda_n}$

Case 2: $\lambda = -\mu^2 < 0$

Arguing in a similar fashion we find that the Green's function associated with (16) for this case is

$$(23) \quad G_n(\beta,\eta) = \frac{2}{\ell} \frac{\sin(\frac{n\pi}{\ell}\xi)}{\mu_n \operatorname{sh}(\mu_n \ell)} \begin{cases} \operatorname{sh}(\mu_n(\ell-\eta)) \operatorname{sh}(\mu_n\beta) & 0<\beta<\eta \\ \operatorname{sh}(\mu_n\eta) \operatorname{sh}(\mu_n(\ell-\beta)) & \eta<\beta<\ell \end{cases}$$

Case 3: $\lambda=0$

In this case the desired Green's function is

$$(24) \qquad G_n(\beta,\eta) = \sqrt{\frac{2}{\ell}}\ \frac{\sin \frac{n\pi}{\ell}\xi}{\ell} \begin{cases} \beta(\hat{\ell}-\eta) & 0<\beta<\eta \\ \eta(\hat{\ell}-\beta) & \eta<\beta<\ell \end{cases}$$

Note that (22) and (23) can be given by the same formula if we take μ_n to be a complex number (either $i|\mu_n|$ or $|\mu_n|$). Also, (24) can be obtained as the limit of either (22) or (23) as $\mu_n \rightarrow 0$. However, it is best to split the expression as above to maximize computational efficiency. The procedure we have followed in representing the Green's function has two advantages over the classical expansion of the Green's function in terms of eignenfunctions of the whole problem (as used in Lang and Staelin 1982b). First, it reduces the expression of the Green's function to a single sum instead of a double sum as in the classical representation. Second, the series has a strong convergence property. Since there is only a finite number of negative eigenvalues, the main part of the series is given by (22) which can be expressed in terms of exponentials. This series converges exponentially fast. In fact, the bilinear eigenfunction expansion for the Green's function fails in this case since $\lambda_0 = 0$ is an eigenvalue for an infinite number of w's, and the series diverges in these cases. The complete expression for the Green's function is

$$K(\alpha,\beta,\xi,\eta) = \sum_{n=1}^{[n_0]} \frac{2}{\ell} \frac{\sin(\frac{n\pi}{\ell}\xi) \quad \sin(\frac{n\pi}{\ell}\alpha)}{\mu_n \sin\mu_n \hat{\ell}} \begin{cases} \sin(\mu_n(\hat{\ell}-\eta)) \sin(\mu_n\beta) \\ \sin(\mu_n\eta) \sin(\mu_n(\hat{\ell}-\beta)) \end{cases}$$

$$(25) \qquad + \epsilon[n_0] \frac{2}{\ell} \frac{\sin(\frac{n\pi}{\ell}\xi) \sin(\frac{n\pi}{\ell}\alpha)}{\hat{\ell}} \begin{cases} \beta(\hat{\ell}-\eta) \\ \eta(\hat{\ell}-\beta) \end{cases}$$

$$+ \sum_{n=[n_0]+1}^{\infty} \frac{\omega^2}{\ell} \frac{\sin(\frac{n\pi\xi}{\ell}) \sin(\frac{n\pi\alpha}{\ell})}{\mu_n \operatorname{sh}(\mu_n\ell)} \frac{\operatorname{sh}(\mu_n(\hat{\ell}-\eta)) \operatorname{sh}(\mu_n\beta) \quad 0<\beta<\eta}{\operatorname{sh}(\mu_n\eta) \operatorname{sh}(\mu_n(\ell-\beta)) \quad \eta<\beta<\hat{\ell}}$$

Here we have used the notation

$$(26) \quad n_0 = \sqrt{\frac{\gamma^2+\omega^2}{a_\alpha \pi^2}}$$

and $[n_0]$ is the integer strictly less that n_0, including the case when n_0 is an integer. Also,

$$(27) \quad \epsilon[n_0] = \begin{cases} 0 & n_0 \text{ not integer} \\ 1 & n_0 \text{ integer} \end{cases}$$

and

$$(28) \quad \mu_n = \sqrt{|\lambda_n|} = \sqrt{\left|\frac{a_\alpha}{a_\beta} \frac{n^2\pi^2}{\ell} - \frac{\gamma^2+\omega^2}{a_\beta}\right|} \quad n=1,2,\ldots$$

Finally, note that the Green's function has the symmetry property

$$(29) \quad K(\alpha,\beta,\xi,\eta) = K(\xi,\eta,\alpha,\beta) \qquad \text{for } \eta<\beta<\hat{\ell}$$

4.2 Transfer Function of the Controlled Membrane

Recall the reduced form of the model (7). Using the definition of the Green's function and the symmetry property (29) we can express the transfer function of the system as

$$(30) \qquad H(\alpha,\beta,\omega) = \iint_{\Omega} a_{\beta} \; K_{\omega}(\alpha,\beta,\xi,\eta) \quad u(\xi,\eta,\omega) \; d\xi \, d\eta = \langle K,u \rangle$$

Now consider the piecewise constant subdivision of the rectangular area of the membrane as shown in Figure 4.1

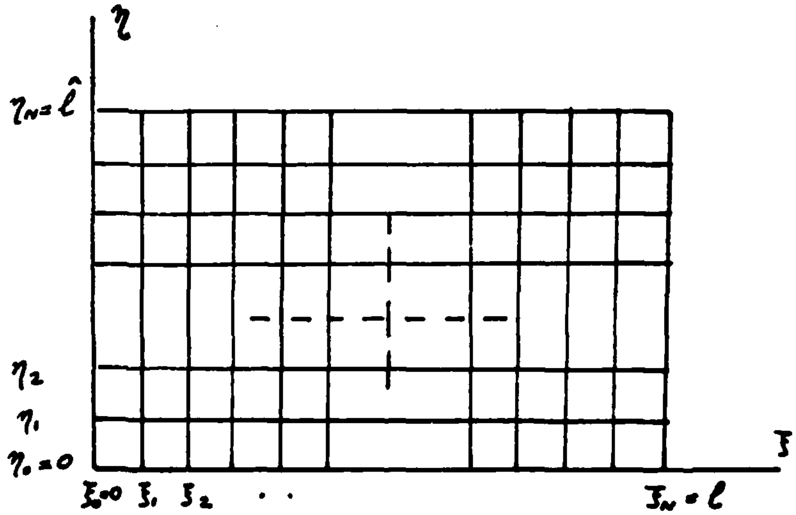


Fig. 4.1 Membrane subdivision.

Assume that the control input is constant over each small rectangle

$$(31) \quad \Omega_{ij} = [\xi_i, \xi_{i+1}] \times [\eta_j, \eta_{j+1}] \qquad i, j, = 1, \ldots, N$$

That is,

$$(32) \quad u(\xi,\eta,\omega) = u_{ij}(\omega) \text{ for } \begin{cases} \xi_i \le \xi < \xi_{i+1} \\ \eta_j \le \eta < \eta_{i+1} \end{cases}$$

Using this, the transfer function can be expressed

$$H(\alpha,\beta,\omega) = \iint_{\Omega} a_\beta K_\omega(\alpha,\beta,\xi,\eta)\, u(\xi,\eta,\omega)\, d\xi\, d\eta$$

$$(33) \quad = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ \beta<\eta_j}}^{N} a_\beta u_{ij}(\omega) \iint_{\Omega_{ij}} K_\omega(\alpha,\beta,\xi,\eta)\, d\xi\, d\eta$$

To compute the integral over Ω_{ij} it is necessary to distinguish three cases: $\beta < \eta_j$, $\beta > \eta_{j+1}$ and $\beta \epsilon [\eta_j, \eta_{j+1}]$.

Case 1: $\beta < \eta_j$

Substituting the expression for the Green's function (25) into (33), we obtain

$$\iint_{\Omega_{ij}} K^\omega(\alpha,\beta,\xi,\eta)\, d\xi d\eta$$

$$(34) \quad = \iint_{\Omega_{ij}} \sum_{\substack{n=1 \\ [n_0]-\epsilon}} \frac{2}{\ell} \frac{\sin(\frac{n\pi}{\ell}\xi)\sin(\frac{n\pi}{\ell}\alpha)\sin(\mu_n(\hat{\ell}\eta))\sin(\mu_n\beta)}{\mu_n \sin(\mu n \hat{\ell})} d\xi d\eta$$

$$+ \iint_{\Omega_{ij}} \epsilon \frac{2}{\ell} \frac{\sin(\frac{n_0\pi}{\ell}\xi)\sin(\frac{n_0\pi}{\ell}\alpha)\beta(\hat{\ell}-\eta)}{\hat{\ell}} d\xi d\eta$$

$$+ \iint_{\Omega_{ij}} \sum_{n=[n_0]+1}^{\infty} \frac{2}{\ell} \frac{\sin(\frac{n\pi}{\ell}\xi)\ \sin(\frac{n\pi}{\ell}\alpha)}{\mu_n \ \mathrm{sh}(\mu_n \hat{\ell})} \ \mathrm{sh}(\mu_n(\hat{\ell}-\eta))\ \mathrm{sh}(\mu_n \beta)\ d\xi d\eta$$

Each of the integrals in (34) may be computed in closed form. We shall omit the details and state the final result

$$\iint_{\Omega_{ij}} K_\omega(\alpha,\beta,\xi,\eta)\,d\xi d\eta$$

$$= \sum_{n=1}^{[n_0]-\varepsilon} \frac{2\sin(\frac{n\pi}{\ell}\alpha)\sin(\mu_n\beta)}{n\pi\mu_n^2 \ \sin(\mu_n \hat{\ell})} \left[\cos(\frac{n\pi}{\ell}\xi_{i+1})\right]\left[\begin{matrix}\cos(\mu_n(\hat{\ell}-\eta_{j+1})) \\ -\cos(\mu_n(\hat{\ell}-\eta_j))\end{matrix}\right]$$

$$(35) \quad + \frac{2\varepsilon}{n_0\pi\ell} \ \sin(\frac{n_0\pi}{\ell}\xi_i)\left[-\cos(\frac{n_0\pi}{\ell}\xi_{i+1})\right]\left[\hat{\ell}(\eta_{j+1}-\eta_j) - \frac{\eta_{j+1}^2-\eta_j^2}{2}\right]$$

$$+ \varphi \sum_{n=[n_0]+1}^{\infty} \frac{2\sin(\frac{n\pi}{\ell}\alpha)\ \ \mathrm{sh}(\mu_n\beta)}{n\pi\mu_n^2\ \mathrm{sh}(\mu_n\hat{\ell})} \left[\cos(\frac{n\pi}{\ell}\xi_i)-\cos(\frac{n\pi}{\ell}\xi_{i+1})\right]\left[\begin{matrix}\mathrm{ch}(\mu_n(\hat{\ell}-\eta_j)) \\ -\mathrm{ch}(\mu_n(\ell-\eta_{j+1}))\end{matrix}\right]$$

Case 2: $\beta > \eta_{j+1}$

The final result in this case is

$$\iint_{\Omega_{ij}} K_\omega(\alpha,\beta,\xi,\eta)\ d\xi d\eta$$

$$(36) \quad = \sum_{n=1}^{[n_0]-\varepsilon} \frac{2\sin(\frac{n\pi}{\ell}\alpha)\ \sin(\mu_n(\hat{\ell}-\beta))}{n\pi\mu_n^2\ \sin(\mu_n\hat{\ell})}\left[\cos(\frac{n\pi}{\ell}\xi_i)-\cos(\frac{n\pi}{\ell}\xi_{i+1})\right]\left[\begin{matrix}\cos(\mu_n\eta_j) \\ -\cos(\mu_n\eta_{j+1})\end{matrix}\right]$$

$$+ \frac{2\varepsilon}{n_0\pi\ell}\ \sin(\frac{n_0\pi}{\ell}\alpha)\ (\hat{\ell}-\beta)\left[\cos(\frac{n_0\pi}{\ell}\xi_i) - \cos(\frac{n_0\pi}{\ell}\xi_{i+1})\right]\frac{\eta_{j+1}^2-\eta_j^2}{2}$$

$$\sum_{n=[n_0]+1}^{\infty} \frac{2\sin(\frac{n\pi}{\ell}\alpha)\ \mathrm{sh}(\mu_n(\hat{\ell}-\beta))}{n\pi\mu_n^2\ \mathrm{sh}(\mu_n\hat{\ell})}\left[\cos(\frac{n\pi}{\ell}\xi_i)-\cos(\frac{n\pi}{\ell}\xi_{i+1})\right]\left[\begin{matrix}\mathrm{ch}(\mu_n\eta_{j+1}) \\ -\mathrm{ch}(\mu_n\eta_j)\end{matrix}\right]$$

Case 3: $\mu_j \le \beta \le \mu_{j+1}$

This case may be treated by substituting β for η_j in Case 1 or for η_{j+1} in Case 2. The final result is

$$\iint_{\Omega_{ij}} K_{\omega}(\alpha, \beta, \xi, \eta)\, d\xi d\eta = \sum_{n=1}^{[n_0]} -\frac{2\sin(\frac{n\pi}{\ell}\alpha)}{n\pi\mu_n^2 \sin(\mu_n \hat{\ell})}\left[\cos(\frac{n\pi}{\ell}\xi_i)-\cos(\frac{n\pi}{\ell}\xi_{i+1})\right]\left[\sin(\mu_n(\hat{\ell}-\Delta))\right.$$

(37)

$$\times\{\cos(\mu_n\eta_j)-\cos(\mu_n\beta) + \sin(\mu_n\beta)\{\cos(\mu_n(\hat{\ell}-\eta_{j+1}))-\cos(\mu_n(\hat{\ell}-\beta))$$

$$+\frac{2\varepsilon\sin(\frac{n_0\pi}{\ell}\alpha)}{n_0\pi\ell}\left[\cos(\frac{n\pi}{\ell}\xi_i-\cos(\frac{n\pi}{\ell}\xi_{i+1})\right]\times\left[\operatorname{sh}(\mu_n(\hat{\ell}-\beta))\{\operatorname{ch}(\mu_n\beta)-\operatorname{ch}(\mu_n\eta_j)\right.$$

$$\left.+\operatorname{sh}(\mu_n\beta)\operatorname{ch}(\mu_n(\hat{\ell}-\beta)-\operatorname{ch}(\mu_n(\hat{\ell}-\eta_{j+1}))\}\right] \sum_{n=[n_0]+1}^{\infty}\frac{2\sin(\frac{n\pi}{\ell}\alpha)}{n\pi\mu_n^2 \operatorname{sh}(\mu_n\hat{\ell})}$$

$$\cos(\frac{n\pi}{\ell}\xi_i-\cos(\frac{n\pi}{\ell}\xi_{i+1})\times\left]\operatorname{sh}(\mu_n(\hat{\ell}-\beta))\{\operatorname{ch}(\mu_n\beta)-\operatorname{ch}(\mu_n\eta_j)\right.$$

$$\left.+\operatorname{sh}(\mu_n\beta)\operatorname{ch}(\mu_n(\hat{\ell}-\beta)-\operatorname{ch}(\mu_n(\hat{\ell}-\eta_{j+1}))\}\right]$$

4.3 Solution of the elliptic system using the discretized Green's function

We can use the discretized Green's function as the basis for an effective algorithm to compute approximate solutions to the system. The algorithm is more efficient than direct numerical integration of the partial differential equation. In general terms the procedure is as follows: Consider the complex elliptic PDE

$$a\frac{\partial^2 H}{\partial x^2} + b\frac{\partial^2 H}{\partial y^2} + pH = -bH \quad \text{in } \Omega$$

(38)

$$H = 0 \text{ on } \partial\Omega,\ \Omega = [0,L] \times [0,\hat{L}]$$

where a and b are as in (7) and p is a complex parameter (which will depend on the frequency ω). Let

$$n_0 = [pL^2/a\pi^2]^{\frac{1}{2}} \qquad \mu_n = [\frac{a}{b}\frac{n^2\pi^2}{L^2} - \frac{p}{b}]^{\frac{1}{2}}$$

(39)

$$\varepsilon_0(n_0) = \begin{cases} 1 & n_0 \text{ integer} \\ 0 & \text{otherwise} \end{cases}$$

Assume that the domain $\Omega = [0,L] \times [0,\hat{L}]$ is subdivided into small rectangles as in Figure 4.1

(40) $\Omega_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ i=1,...,N, j=1,...,M

Let $h = L/N$ and $k = \hat{L}/M$. Assume that the control is constant in space over the rectangle Ω_{ij} and defined by its value at the center. That is,

(41) $$u(x,y,p) = u_{ij}(p) = u\left(\frac{x_i+x_{i+1}}{2}, \frac{y_j+y_{j+1}}{2}, p\right)$$

Let $G_{ij}(x,y,p)$ be the average Green function over Ω_{ij}

(42) $$G_{ij}(x,y,p) = \iint_{\Omega_{ij}} K(x,y,\xi,\eta,p)\, d\xi d\eta$$

Then the solution to the complex PDE is given by

(43) $$H(x,y,p) = \sum_{i=1}^{N} \sum_{j=1}^{M} bu_{ij}(p)\, G_{ij}(x,y,p)$$

We introduce the change of notation

(44)
$$G_{ij}(x,y,p) = G(x,x_i,x_{i+1},y,y_j,y_{j+1},p)$$
$$u_{ij}(p) = u(x_i,x_{i+1},y_j,y_{j+1},p)$$

which will be useful in programming the algorithm. The function $G_{ij}(x,y,p)$ will be given by the following expressions:

(45a) $$G^{I}_{ij}(x,y,p) = -\{G_1(x,x_i,x_{i+1}, \hat{L}-y, \hat{L}-y_j, \hat{L}-y_{j+1}, p)+G_2(-)\},\ y < y_j$$

(45b) $$G^{II}(x,y,p) = G_1(x,x_i,x_{i+1},y, y_j,y_{j+1},p) + G_2(-),\ y_j < y$$

(45c) $$G^{III}(x,y,p) = G^{II}(x,x_i x_{i+1},y,y_j,y,p) + G^{I}(x,x_i,x_{i+1},\hat{L}-y,\ \hat{L}-y_{j+1},\hat{L}-y,p)$$

$$y_i \le y \le y_{j+1}$$

where

(46a)
$$G_1(-) = \{ \sum_{\substack{n=1\\ n\neq n_0}}^{\infty} \frac{2}{n\pi\mu_n^2 \, \mathrm{sh}(\mu_n \hat{L})} \sin\left(\frac{n\pi x}{L}\right) \mathrm{sh}\,[\mu_n(\hat{L}-y)] \}$$

$$\times \{\cos\left(\frac{n\pi x_i}{L}\right) - \cos\left(\frac{n\pi x_{i+1}}{L}\right)\} \times \{\mathrm{ch}(\mu_n y_{j+1}) - \mathrm{ch}(\mu_n y_j)\}$$

(46b) $$G_2(-) = \varepsilon(n_C)\frac{2}{n_0\pi L}\sin\left(\frac{n_0\pi x}{L}\right)(\hat{L}-y)\cdot[\cos\left(\frac{n_0\pi}{L}x_i\right) - \cos\left(\frac{n_0\pi}{L}x_{i+1}\right)]$$

$$\times\frac{1}{2}(y_{j+1}^2 - y_i^2)$$

The computational algorithm based on this representation of the solution is given in Figure 4.2.

4.4 Eigenfunction expansion of the solution of the generalized wave equation

In this subsection we shall use an eigenfunction expansion to solve the system (1) as an evolution system in time. Using the exponential transformation (2) we can rewrite the system as

(47)
$$\frac{\partial^2 g}{\partial t^2} = a\frac{\partial^2 g}{\partial x^2} + b\frac{\partial^2 g}{\partial y^2} + dg + bu \quad \text{in } \Omega$$

$$g=0 \text{ on } \partial\Omega,\ g(x,y,t_0) = g_0(x,y),\ g_t(x,y,t_0) = g_1(x,y)$$

Here we use general boundary conditions since we want to use the solution to simulate the response of the membrane system to inputs based on its state at a given time. Our solution is expressed in terms of an eigenfunction expansion. Let $\phi_{mn}(x,y)$ be the eigenfunctions of the operator in (47) and suppose that the data and the control function have the form

$$u(x,y,t) = \sum_{m,n} u_{mn}(t)\,\phi_{mn}(x,y)$$

(48)
$$g_0(x,y,t_0) = \sum_{m,n} r_{mn}\,\phi_{mn}(x,y) \quad g_1(x,y,t_0) = \sum_{m,n} v_{mn}\,\phi_{mn}(x,y)$$

We look for the solution in the form

$$g(x,y,t) = \sum_{m,n} \alpha_{mn}(t)\,\phi_{mn}(x,y) \tag{49}$$

This leads to an eigenvalue problem

(50)
$$a\frac{\partial^2\phi_{mn}}{\partial x^2} + b\frac{\partial^2\phi_{mn}}{\partial y^2} + d\,\phi_{mn} = \lambda_{mn}\phi_{mn}$$
$$\phi_{mn} = 0 \text{ on } \partial\Omega$$

and an initial value problem

(51)
$$\ddot{\alpha}_{mn}(t) = \lambda_{mn}\,\alpha_{mn}(t) + b\,u_{mn}(t)$$
$$\alpha_{mn}(0) = r_{mn}, \quad \dot{\alpha}_{mn}(0) = v_{mn}$$

A simple calculation determines the eigenfunctions

$$\phi_{mn}(x,y) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{\hat{L}}\right) \tag{52}$$

corresponding to the eigenvalues

$$\lambda_{mn} = d - \left[a\left(\frac{n\pi}{L}\right)^2 + b\left(\frac{m\pi}{\hat{L}}\right)^2\right] \tag{53}$$

The initial value problem has different solution forms depending on the sign of λ_{mn}. Let t_0 be the initial time and let $t_f = t_0 + T$, where T is the sampling period. When $\lambda_{mn} < 0$, we have

$$\alpha_{mn}(t_f) = r_{mn} \cos(\mu_{mn}T) + \frac{v_{mn}}{\mu_{mn}} \sin(\mu_{mn}T) + \frac{b}{\mu_{mn}} \cdot u_{mn} \cdot \frac{1-\cos \mu_{mn}T}{\mu_{mn}}$$

(54a)

$$\dot{\alpha}_{mn}(t_f) = - r_{mn}\, \mu_{mn} \sin(\mu_{mn}T) + v_{mn} \cos(\mu_{mn}T) + bu_{mn} \frac{\sin \mu_{mn} T}{\mu_{mn}}$$

When $\lambda_{mn} = 0$, we have

$$\alpha_{mn}(t_f) = r_{mn} + v_{mn}T + bu_{mn} \frac{T^2}{2}$$

(54b)

$$\dot{\alpha}_{mn}(t_f) = v_{mn} + bu_{mn}T$$

When $\lambda_{mn} > 0$, we have

$$\alpha_{mn}(t_f) = r_{mn}\, \mathrm{ch}(\mu_{mn}T) + \frac{v_{mn}}{\mu_{mn}} \mathrm{sh}(\mu_{mn}T) + \frac{b}{\mu_{mn}} u_{mn} \frac{\mathrm{ch}(\mu_{mn}T)-1}{\mu_{mn}}$$

(54c)

$$\dot{\alpha}_{mn}(t_f) = \mu_{mn} r_{mn}\, \mathrm{sh}(\mu_{mn}T) \quad v_{mn}\, \mathrm{ch}(\mu_{mn}T) + bu_{mn} \frac{\mathrm{sh}(\mu_{mn}T)}{\mu_{mn}}$$

These expressions permit us to solve the membrane system from any initial conditions for any control input satisfying the condition that it be constant over the area of the mesh element in Figure 4.1.

4.5 Spectral factorization using the Hilbert transform

Computation of the optimal stabilizing gain based on the Davis - Stenger spectral factorization procedure used for the Euler beam problem proved to be infeasible in treating the two dimensional system. The computational effort was too high to be practical; and it was necessary to redesign the algorithm to achieve greater numerical efficiency. The key idea was to use the Hilbert transform to represent the spectral factors arising in the calculation. This permitted the use of fast and efficient numerical Fourier transform techniques in the gain computation algorithm.

Given a Fourier pair (f,F) in $L^2 \times L^2$

$$(55) \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} F(s)ds, \quad F(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx} f(x)dx$$

the Hilbert transform is defined by

$$(56) \qquad Hf(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(r)}{t-r}\, dr \quad \epsilon\, L^2$$

where the integral is taken in the Cauchy principal value sense. Given a complex function

$$(57) \qquad T(s) = R(s) + iF(s)$$

then

$$(58) \; HT(\omega) = HR(\omega) + iHF(\omega)$$

The inverse Fourier transform of HT satisfies

$$(59)\quad F^{-1}\{HT\} = -i\,\mathrm{sgn}(t)\, F^{-1}\{T\}$$

If we define

$$(60)\quad T_+ = \frac{1}{2}(T - iHT),\quad T_- = \frac{1}{2}(T + iHT)$$

then

$$(61)\quad F^{-1}(T_+) = \begin{cases} 0 & t > 0 \\ F^{-1}(T) & t < 0 \end{cases},\quad F^{-1}(T_-) = \begin{cases} F^{-1}(T) & t > 0 \\ 0 & t < 0 \end{cases}$$

This means that T_+ (respectively T_-) is the projection of T onto the space L^2_+ (respectively L^2_-) with the obvious intepretation of L^2_+ and L^2_-.

Now consider a function $f(x)$ satisfying the condition

$$(62)\quad \lim_{|x| \to \infty} f(x) = 1$$

if $\phi(x) = f(x) - 1$, then $\phi \in L^2$. Applying the previous results to the function

$$(63)\quad \psi(x) = \log[f(x)]$$

(with the proper choice for the branch of log), then

$$(64)\quad f(x) = f^+(x) \cdot f^-(x)$$

where f^+ and f^- are the causal and anti-causal spectral factors, respectively. For instance,

(65) $$f^{-}(x) = f(x)\exp\{-\frac{i}{2}\, H\log f(x)\}$$

This expression is used directly in the computation of the optimal stabilizing feedback gain by spectral factorization.

4.6 Gain computations

In abstract form the control system (1) takes the form

(66) $$\dot{x}(t) = Ax + Bu$$
$$y(t) = Cx \quad , \; x(0) = x_0$$

The transfer function is

(67) $$G(i\omega) = CR(i\omega;A)\,B$$

where $R(i\omega;A)$ is the resolvent operator

(68) $$R(i\omega;A) = (i\omega I - A)^{-1}$$

If we compute the spectral factor of $I + G^{*}G$

(69) $$F^{-}(i\omega)\, F^{+}(i\omega) = (I + G^{*}G)(i\omega)$$

then the gain is given by

(70) $$[B^{*}K](x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (F^{-}(i\omega))^{-1}\, G^{*}(i\omega)CR(i\omega,A)\, x_0$$

In the present case the components are

$$(71)\quad A=\begin{bmatrix} 0 & 1 \\ a\partial_{xx}+b\partial_{yy}+d & c \end{bmatrix},\quad b=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

And we shall take

$$(72)\quad C=\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \in \mathcal{L}(L^2 \oplus L^2)$$

which allows for weights on both the displacement and the velocity.

The resolvent operator is computed by solving the system

$$(73)\quad \begin{pmatrix} u \\ v \end{pmatrix} = (sI-A)\begin{pmatrix} f \\ g \end{pmatrix}$$

which leads to the system

$$(74)\quad \begin{aligned} & a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial y^2} - (s^2-cs)\, f = (cu-v) - su \\ & f=0 \text{ on } \partial\Omega,\ g=sf-u \end{aligned}$$

The associated transfer function is

$$(75)\quad G(i\omega) = \begin{bmatrix} k_1\, g_1(i\omega) \\ k_2\, g_2(i\omega) \end{bmatrix}$$

where

$$(76) \quad a \frac{\partial^2 g_1}{\partial x^2} + b \frac{\partial^2 g_1}{\partial y^2} + (\omega^2 + ic)\, g_1 = -1$$

$$g_1 \text{ on } \partial\Omega , \quad g_2 = i\, g_1$$

Therefore, if we define $H(i\omega) = (I + G^* G)(i\omega)$ we obtain

$$(77) \qquad H(i\omega) = 1 + (K_1^2 + \omega^2 K_2^2)\; g_1(j\omega)$$

Let the state of the system be defined by

$$(78) \quad x(t) = \begin{bmatrix} h \\ h_t \end{bmatrix}$$

and let

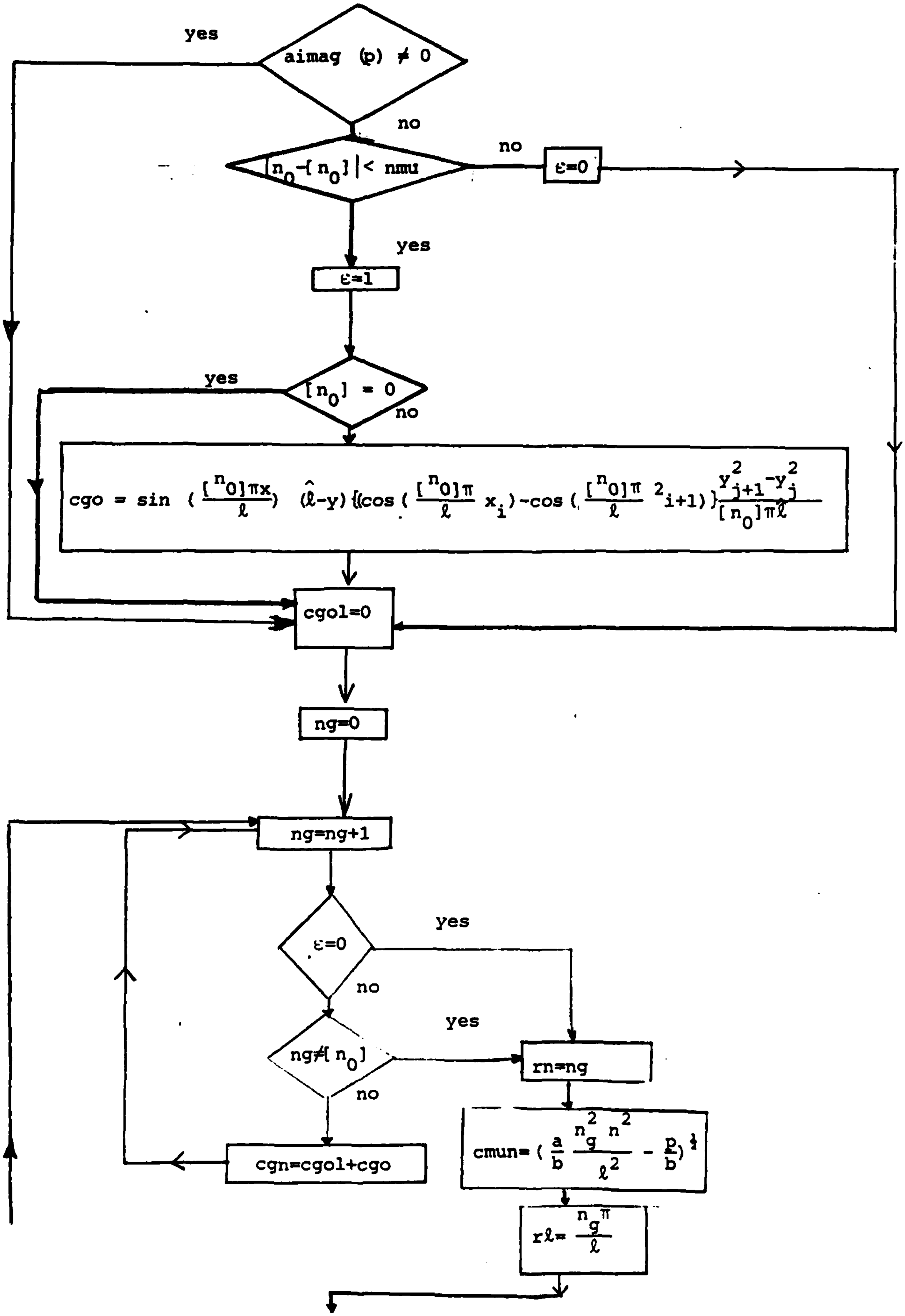
$$(79) \quad \begin{bmatrix} f \\ g \end{bmatrix} = R(i\omega; A) \begin{bmatrix} h \\ h_t \end{bmatrix}$$

and let g_1 satisfy the complex PDE (76). Then the stabilizing feedback gains are given by

$$(80) \quad B^*K \begin{bmatrix} h \\ h_t \end{bmatrix} = \int_{-\infty}^{\infty} \Big[F^-(i\omega)^{-1} g_1^* (i\omega) K(i\omega) (K_1^2 + \omega^2 K_2^2) + i\omega K_1^2 \; F^-(i\omega) \Big]^{-1} g_1^* h \, d\omega$$

4.7 Software development and control system performance

Fortran code has been developed to implement the control gain computation (80) using the Hilbert transform representation (65) for the anti-causal spectral factor of the return matrix. The code includes the simulation algorithm for the system response to the control as shown in Figure 4.2. Testing of the code has been carried out for a few values of the weights in (72). The results indicate that the stabilization of the system is significantly enhanced by the control action. Since we take the damping in (1) to be positive, the system is stable for small enough bias voltages. Modest values of the control weights (10 on a normalized scale), produce a 20% improvement in the settling time of the system. Extensive testing will require an upgrading of the numerical routines in the code to produce faster solutions of the system.



aimag (p) ≠ 0
yes
no
|n0-[n0]| < nmu
no
ε=0
yes
ε=1
[n0] = 0
yes
no
cgo = sin ([n0]πx/ℓ) (ℓ̂-y) {(cos ([n0]π/ℓ xi) - cos ([n0]π/ℓ x i+1)} (y²j+1 - y²j)/([n0]πℓ)
cgol=0
ng=0
ng=ng+1
ε=0
yes
no
ng≠[n0]
yes
no
rn=ng
cgn=cgol+cgo
cmun= (a/b ng² n²/ℓ² - p/b)^½
rℓ= ngπ/ℓ

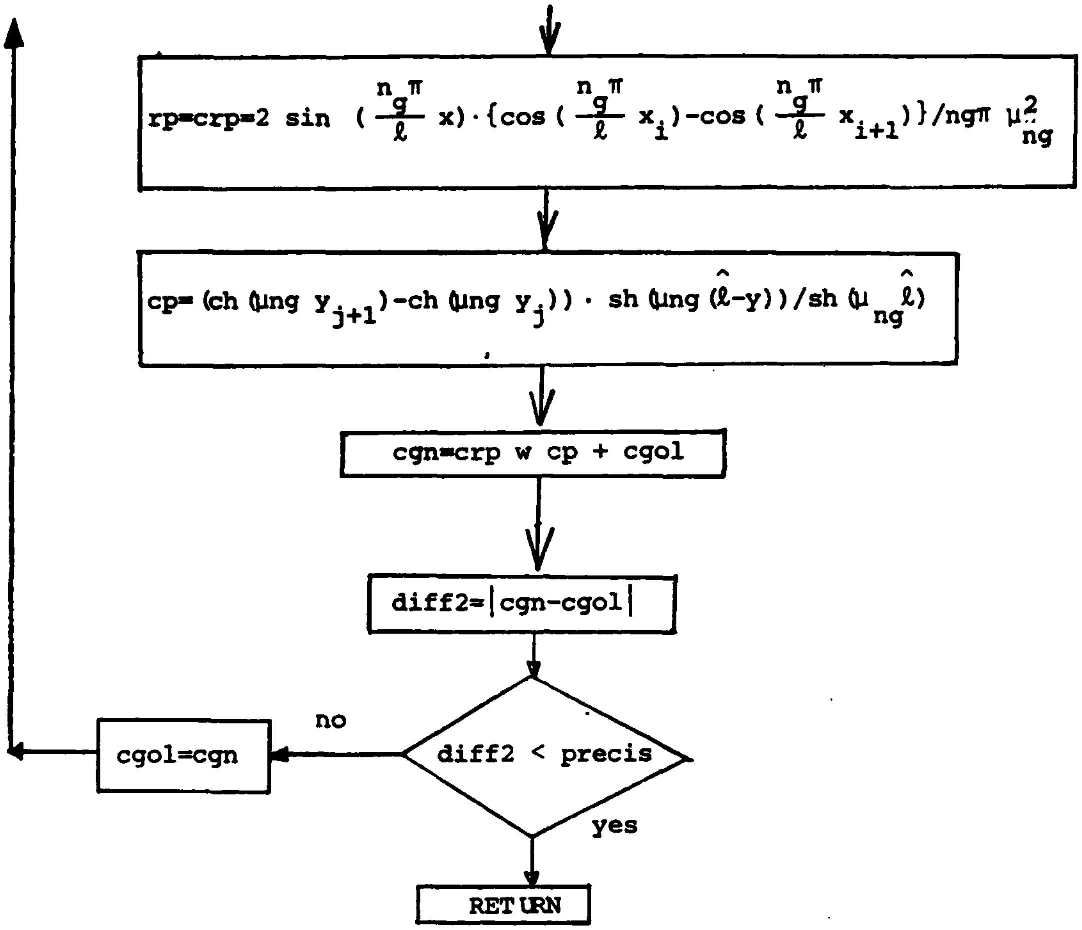


Figure 4.2 Algorithm for control gain computation and simulation

Part II:

Effective Parameter Models of Heterogenous Structures

5. Homogenization of Regular Structures

It is now generally accepted that large, low mass lattice structures, e.g., trusses, are natural for space applications. Their large size and repetitive infrastructure require special techniques for structural analysis to cope with the large number of degrees of freedom. As Noor, Anderson, and Greene (1978) point out, continuum models provide a simple means for comparing structural characteristics of lattices with different configurations, and they are effective in representing macroscopic vibrational modes and structural response due to temperature and load inputs. Our approach to the construction of such models is presented in this section. In the next two sections we consider the problems of control and state estimation in combination with the construction of continuum models. We shall begin with a few remarks on related work on continuum models in the recent structural mechanics literature.

Noor, et. al. (1978) use an energy method to derive a continuum approximation for trusses with triangular cross sections in which the modal displacements of the truss are related to a linearly varying displacement field for an equivalent bar. In (Dean and Tauber 1959) and (Renton 1969), exact analytical expressions for the solutions of trusses under load were derived using finite difference calculus. By expressing the difference operators in terms of Taylor's series Renton (1970) was able to derive continuum approximations to the finite difference equations resulting in expressions for equivalent plate stiffnesses, for example. In a recent paper Renton (1984) used this approach to give equivalent beam properties for trusses, and this complements the earlier work of Noor, Anderson and Greene (1978) and Nayfeh and Hefzy (1978). (See also (Anderson 1981).)

In these papers a continuum model is associated with the original (lattice) structure by averaging the parameters of the lattice over some natural volume (e.g. of a "cell" of the structure) and identifying the averaged parameter value (mass density, stress tensor, etc.) with the corresponding distributed parameter in the continuum model. A specific form for the continuum model is postulated at the outset of the analysis; e.g., a truss with lattice structure will be approximated by a beam, with the beam dynamical representation assumed in advance. While this approach has an appealing directness and simplicity, it has some problems.

First, it is very easy to construct an example in which the "approximate model" obtained by averaging the parameters over a cell is not a correct approximation to the system behavior. This is done in subsection 5.1. Second, the averaging method (averaging the parameters over space) does not apply in a straightforward way to systems with a random structure, since the appropriate averaging procedure in this case may not be obvious. Third, the method cannot be naturally imbedded in an optimization procedure; and controls and state estimates based on the averaged model may not be accurate reflections of controls and state estimates derived in the course of a unified optimization - averaging procedure. The method does not provide a systematic way of estimating the degree of suboptimality of controls and state estimates computed from the idealized model.

In this work we use a totally different technique called homogenization from the mathematical theory of asymptotic analysis to approximate the dynamics of structures with a repeating cellular structure. Homogenization produces the distributed model as a consequence of an asymptotic analysis carried out on a rescaled version of the physical system model.

Unlike the averaging method, homogenization can be used in combination with optimization procedures; and it can yield systematic estimates for the degree of suboptimality of controls and estimators derived from idealized models. Results to this effect are given in sections 6 and 7. While our results are preliminary, they nevertheless demonstrate the feasibility of the method; and they suggest its potential in the analysis of structures of realistic complexity.

In this section, we first give an example illustrating some of the subtleties of homogenization; then we discuss homogenization for abstract hyperbolic systems; then we illustrate the applications of homogenization theory by deriving a diffusion approximation for the thermal conductivity of a (random) lattice structure. In the final example we derive a homogenized representation for the dynamics of a lattice structure undergoing transverse deflections. We show that the behavior of the lattice is well approximated by the Timenshenko beam equation; and we show that this equation arises naturally as the limit of the lattice dynamics when the density of the lattice structure goes to infinity in a well defined way. The mathematical analysis used in the derivations is based on the book (Bensoussan, Lions, and Papanicolaou 1978) and the paper (Kunnemann 1983).*

5.1 A one-dimensional example

From (Bensoussan, Lions and Papanicolaou 1978) we have the following example:

$$(1)\qquad \begin{aligned} -\frac{d}{dx}\left(a^{\varepsilon}(x)\,\frac{du^{\varepsilon}}{dx}\right) &= f(x), \quad x \in (x_0, x_1) \\ u^{\varepsilon}(x_0) = 0 &= u^{\varepsilon}(x_1) \end{aligned}$$

where $a(y)$ is periodic with period y_0, $a(y) \geq \alpha > 0$, and $a^{\varepsilon}(x) = a(y/\varepsilon)$. It is simple to show that

$$(2)\qquad ||u^{\varepsilon}||^2_{H^1} \triangleq \int_{x^2}^{x^1} |u^{\varepsilon}(x)|^2 + \left|\frac{du^{\varepsilon}}{dx}(x)\right|^2 dx \leq c$$

and so, $u^{\varepsilon} \to u$ weakly in the space $\dot{H}^1$. Moreover,

$$(3)\qquad a^{\varepsilon} \to M(a) \triangleq \frac{1}{y_0}\int_0^{y_0} a(y)\, dy$$

and it is natural to suppose that $u^{\varepsilon} \to u$ with the limit defined by

$$(4)\qquad \begin{aligned} -\frac{d}{dx}\left[M(a)\,\frac{d}{dx}u(x)\right] &= f(x) \qquad x \in (x_0, x_1) \\ u(x_0) = 0 &= u(x_1) \end{aligned}$$

This is untrue in general (Bensoussan, Lions and Papanicolaou

---

*We are grateful to Professor George Papanicolaou for bringing Kunnemann's paper to our attention.

1978, pp. 8-10). The correct limit is given by

$$(5) \quad -\frac{d}{dx}[\bar{a} \frac{d}{dx} u(x)] = f(x),\ x\varepsilon(x_0,x_1),\ u(x_0) = 0 = u(x_1)$$

with $u(x_0)=0=u(x_1)$

$$(6) \quad \bar{a}=M(a^{-1})^{-1}$$

In general, $M(a) > \bar{a}$; and so, the error is identifying the limit, (4) versus (5), is fundamental.

The system (4) corresponds to averaging the parameter $a^{\varepsilon}(x)$ over a natural cell; a procedure similar to that used in (Noor, Anderson and Greene 1978), (Nayfeh and Hefzy 1978) and (Aswani 1982) to define continuum models for lattice structures. As (5) shows, the actual averaging process can be more subtle than one might expect, even for simple problems.

To see how (5) arises, let

$$(7) \quad \xi^{\varepsilon}(x)=a^{\varepsilon}(x)\frac{d}{dx}u^{\varepsilon}(x)$$

Then $\xi^{\varepsilon}(x)$ is bounded in $L^2(x_0,x_1)$ and it satisfies

$$(8) \quad -\frac{d}{dx}\xi^{\varepsilon}(x)=f(x),\ x\ \varepsilon\ (x_0,x_1)$$

One can show that $\xi^{\varepsilon}(x)$ has a strong limit $\xi(x)$ in L^2; so

$$(9) \quad \frac{1}{a^{\varepsilon}}\xi^{\varepsilon} \to M(\frac{1}{a})\ \xi$$

weakly in $L^2(x_0,x_1)$. But

$$(10) \quad \xi^{\varepsilon}/a^{\varepsilon} = \frac{du^{\varepsilon}}{dx}$$

so

$$(11) \quad \frac{du}{dx} = M\left(\frac{1}{a}\right)\xi$$

and since

$$(12) \quad -\frac{d\xi}{dx} = f$$

we have

$$(13) \quad -\frac{d}{dx}\left[\left(M\left(\frac{1}{a}\right)\right)^{-1}\frac{du}{dx}\right]=f$$

and the limit $u^{\varepsilon} \to u$ is weak in $H^1(x_1,x_2)$ (the Hilbert space with norm defined by (2)).

This example illustrates the pitfalls associated with simplistic averaging procedures.

5.2 Homogenization of Wave Equations

As shown in Part I of this report, hyperbolic (wave) equations are the most natural models for the dynamics of flexible structures. General techniques for homogenization of wave equations are available (Bensoussan, Lions and Papanicolaou 1978). The precise form of the homogenization procedure depends on the scaling of the physical model; i.e., the dependence of the system characteristic features on small parameters. Since this is a sensitive modeling issue, we shall discuss it in some detail. It may be necessary to consider several scalings to determine the one most suitable for a given class of flexible structures.

Consider the Klein-Gordon equation

$$(14)\quad \begin{aligned} &u_{tt}(t,x)=\nabla[c^2(x)\nabla u(t,x)] - W(x)u(t,x) \\ &u(0,x)=f(x),\ u_t(0,t)=g(x),\ t>0,\ x\varepsilon R^n \end{aligned}$$

where f and g are smooth functions of compact support, $c^2(x) > 0$ and $W(x) \geq 0$ are smooth. Suppose c(x), W(x), f(x), and g(x) depend on a small parameter $\varepsilon > 0$. For example, suppose c and W vary slowly with x, so we have

$$(15)\quad c^2=c^2(\varepsilon x),\ W=W(\varepsilon x)$$

and suppose

$$(16)\quad f = f^{\varepsilon},\quad g = g^{\varepsilon}$$

Let u be the solution of (14) with (15) (16). The behavior of u^{ε} as $\varepsilon \to 0$ in (14)-(16) is trivial if we consider (t,x) fixed. However, if (t,x) become large as $\varepsilon \to 0$, then an interesting limiting behavior emerges (Bensoussan, Lions and Papanicolaou 1978, Chapter 4).

To see this most clearly, it is necessary to rescale t and x as

$$(17)\quad t' = \varepsilon t,$$

Dropping the primes, this yields

$$(18)\quad \begin{aligned} &u^{\varepsilon}_{tt}(t,x)=\nabla\cdot[c^2(x)\nabla u^{\varepsilon}(t,x)]-\frac{1}{\varepsilon^2}W(x)\,u^{\varepsilon}(t,x) \\ &u^{\varepsilon}(0,x)=f^{\varepsilon},\ u^{\varepsilon}_t(\ ,x)=g^{\varepsilon};\ x\ \varepsilon\ R^n,\ t>0 \end{aligned}$$

Notice that in rescaling a system with slowly varying coefficients, one produces a system with a large coefficients.

To complete the specification of the problem it is necessary to define the dependence of the data f^{ε} and g^{ε} on ε. The various choices strongly influence the final form of the limiting system. Following (Bensoussan et al. 1978), we shall distinguish three classes of data.

Case 1: Low frequency problem

In this case f^{ε} and g^{ε} have asymptotic power series expansions

$$(19)\quad \begin{aligned} f^{\varepsilon}(x) &\sim f_0(x) + \varepsilon f_1(x) + \cdots \\ g^{\varepsilon}(x) &\sim g_0(x) + \varepsilon g_1(x) + \cdots \end{aligned}$$

where the terms in the expansion are smooth functions. To produce a problem with finite energy as $\varepsilon \to 0$, it is necessary to take $f_0(x) = 0$ in (19). (See (Bensoussan et al. 1978 p. 541).) Since the problem is linear, it is not necessary to have the expansions begin with ε^2.

The analysis of this problem is comparatively simple, and the limiting behavior as $\varepsilon \to 0$ is elementary.

Case 2: High frequency wave propagation in a slowly varying medium

In this case the data take the apparently specialized form

$$(20)\quad \begin{aligned} f^{\varepsilon}(x) &\sim \exp[iS(x)/\varepsilon]\,\tilde{f}^{\varepsilon}(x) \\ g^{\varepsilon}(x) &\sim \frac{1}{\varepsilon}\exp[iS(x)/\varepsilon]\,\tilde{g}^{\varepsilon}(x) \end{aligned}$$

where $S(x)$ is real-valued and smooth, and $\tilde{f}^{\varepsilon}$ and $\tilde{g}^{\varepsilon}$ are complex-valued and smooth and have asymptotic expansions like (19). Note that the data in (20) are complex-valued. Since the problem (18) is linear, both the real and imaginary parts of u are solutions.

To understand the physical significance of the second case, suppose the "phase function" $S(x) = k\,x$, where k is a constant vector and dot stands for the inner product in R^2. In this instance the data in (20) are spatially modulated plane waves with rapidly varying phase. As shown in (Bensoussan et al. 1978) virtually all cases of interest (different scalings leading to nontrivial limiting behavior) can be analyzed in terms of this case. For instance it is possible to treat the case of wave propagation in slowly varying media with spatially localized data or the form

$$(21)\qquad \begin{aligned} f^{\varepsilon}(x) &= \varepsilon^{1-n/2} \, f\left(x, \frac{x}{\varepsilon}\right) \\ g^{\varepsilon}(x) &= \varepsilon^{-n/2} \, g\left(x, \frac{x}{\varepsilon}\right) \end{aligned}$$

where f and g are smooth functions of compact support in x and y (x/ε), and $(x,y) \in R^2$. The scaling on the right in (21) is chosen so that the terms are of order one as $\varepsilon \to 0$. Problems with forcing functions and/or inhomogenous boundary conditions can also be treated by essentially the same method.

The treatment of case 2 given (Bensoussan et al. 1978) is based on the ideas of geometric opt...s. The solution u^{ε} is sought in the form

$$u^{\varepsilon}(x,t) = \varepsilon\, e^{i\, S(x,t)/\varepsilon} v^{\varepsilon}(x,t)$$

(22)

$$v^{\varepsilon}(x,t) = v_0(x,t) + \varepsilon v_1(x,t) + \dots$$

Inserting (22) into (18) and equating coefficients of equal powers of ε leads to

$$A_1 v_0 = 0$$

(23) $$A_1 v_1 + A_2 v_0 = 0$$

$$A_1 v_2 + A_2 v_1 + A_3 v_0 = 0$$

where
$$A_1 = -c^2 (\nabla S)^2 + (S_t)^2 - W$$

$$A_2 = -2\, i\, S_t\, \partial_t\, \nabla \cdot (c^2 \nabla s) + ic^2 \nabla S \cdot \nabla - i\, S_{tt}$$

(24)

$$A_3 = \nabla \cdot (c^2 \nabla) - \partial_t^2$$

The analysis of these equations leads to the reduced model for the system behavior.

From the first expression in (23) we see that for v_0 not to be identically zero we must have

(25) $$S_t \pm [\, c^2(x)\, (\nabla S)^2 + W]^{1/2} = 0$$

This is the eikonal (or Hamilton-Jacobi) equation, a nonlinear first order PDE which controls the evolution of the phase function. It may be solved in terms of a system of nonlinear ODE's (Hamilton's equations) for the "rays" and "momenta" associated with the propagation of energy by the system. (See (Bensoussan et al. 1978, The choice (25) for the phase makes the operator A_1 identically zero. Using this in the second equation in (3) leads to the "transport equation"

$$(26) \quad A_2 v_0 = 0$$

Its analysis leads to expressions for the propagation of energy in the system.

The case of spatially localized data (21) may be treated using the same techniques in combination with the method of multiple scales. We shall not develop the analysis of these general systems in more detail. Rather, we shall turn our attention to the construction of continuum models for lattice structures.

5.3 Continuum approximations for lattice structures

In this section we shall apply homogenization and the associated asymptotic analysis to derive continuum approximations for two different types of problems. In the first case we show that the problem of thermal energy conduction in a lattice can be well approximated by a diffusion process in the macroscopic scale. In the second case we show that the in-plane macroscopic (2 dimensional) motions of a simplified truss model can be well approximated by the Timoshenko beam system.

5.3.1 Effective conductivity of a periodic lattice

A. Problem definition

Let $Z = \{0,\pm 1,\pm 2,\ldots\}$ and $Z^d = Z \times \ldots \times Z$ (d times) be a d-dimensional lattice. Let $\varepsilon > 0$ be a number small relative to 1. We want to describe the effective conduction of thermal energy on the ε-spaced lattice εZ^d. Let $e_i = (0,\ldots 0,1,0,\ldots 0)^T$ with 1 in the ith position, $i = 1,2,\ldots,d$. If x is a point in Z^d, then $x \pm \varepsilon e_i$, $1 \leq i \leq d$, are the nearest neighbors of x. Let $a_{i\pm}(x)$, $x \in Z^d$, $1 \leq i \leq d$, be the two functions defined on the lattice, and assume

(26a) $a_i(x) := a_{i+}(x) = a_{i-}(x+e_i)$, $x \in Z^d$, $1 < i < d$

(26b) $0 < A \leq a_i(x) \leq B < \infty$ $x \in Z^d$, $1 \leq i \leq d$

(26c) $a_i(x)$ is periodic with period $\ell \geq 1$ in each direction,* $1 \leq i \leq d$.

Next let

(27) $a^{\varepsilon}_{i\pm}(x) = a_{i\pm}(x/\varepsilon)$, $x \in \varepsilon Z^d$, $1 \leq i \leq d$

Equation (26b) means that the conduction process is reversible and that the conductivity $a^{\varepsilon}_i(x)$ is a "bond conductivity", i.e., independent of the direction in which the bond $(x, x + e_i)$ is used by the process. Equation (27) means that the configuration of bond conductivities $a_{i+}(\cdot)$ on εZ^d is simply $a_{i+}(\cdot)$ on Z^d "viewed from a distance." Assumption (26c) imposes a regularity condition on the

*The period may be different in different directions.

physics of the conduction process. An assumption like this is essential for existence of a limit as $\varepsilon \to 0$. In one dimension the situation is illustrated in Figure 5.1. A system similar to this was treated by Kunnemann (1983) with random bond conductivities. Ergodicity replaced periodicity in (Kunnemann 1983).

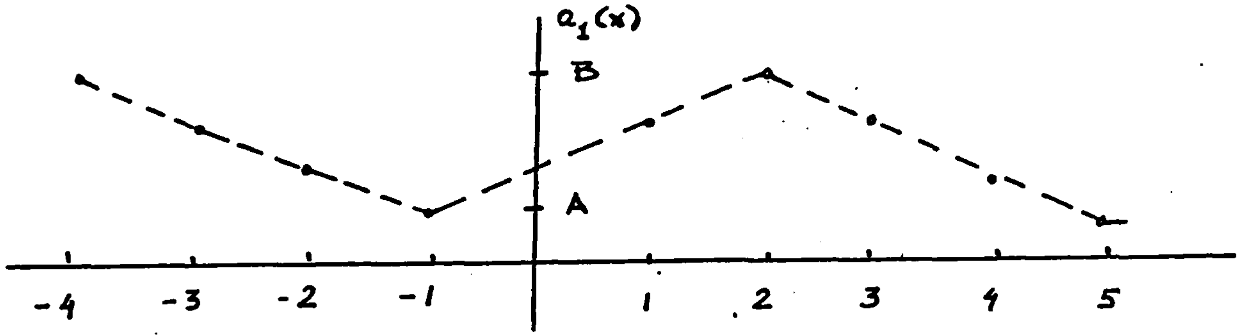


Figure 5.1.a. Conductivity on unscaled lattice with period $\ell=6$.

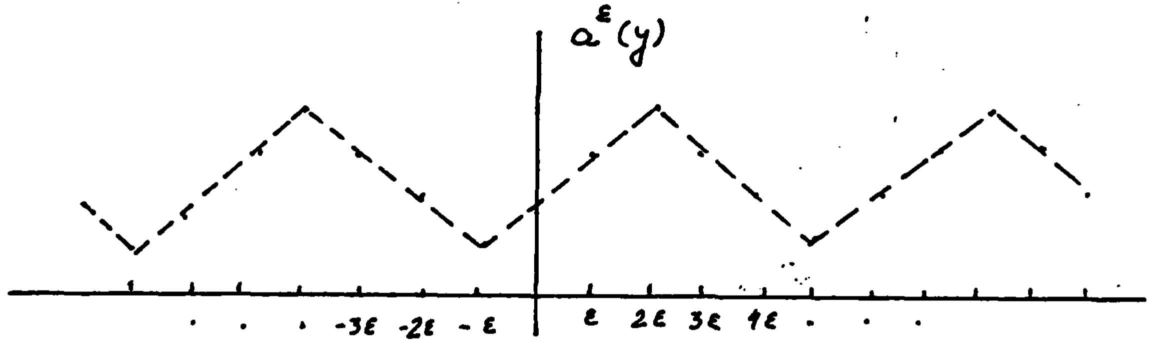


Figure 5.1.b. Conductivity on ε-scaled lattice, $y = \varepsilon x$, with $x \in Z$ and period $\varepsilon\ell = 6$.

One can associate with this system a random (jump) process $\{X^\varepsilon(t,x),\ t \geq 0,\ x \in \varepsilon Z^\delta\}$ on the ε-spaced lattice*. In effect, as $\varepsilon \to 0$, $\{X^\varepsilon\}$ converges to a Brownian motion on the lattice; and the main result of the analysis is an expression for the diffusion matrix

*Definition of this process is not necessary for the analysis, but it bolsters the intuition.

$Q: = [q_{ij};\ i,j = 1,2,\ldots,d]$ of this process. This matrix describes the macroscopic diffusion of thermal energy in the system. It is the effective conductivity.

We shall carry out the asymptotic analysis of this system in the limit as $\varepsilon \to 0$ using the theory of homogenization. Let

$$(28)\quad \begin{aligned} (\nabla_i^{\varepsilon-}u)(x) &:= \frac{1}{\varepsilon}[u(x-\varepsilon e_i)-u(x)] \\ (\nabla_i^{\varepsilon+}u)(x) &:= \frac{1}{\varepsilon}[u(x+\varepsilon e_i)-u(x)] \\ & x\in\varepsilon Z^d,\ 1 \le i \le d, \end{aligned}$$

for any u square summable on Z^d or square integrable on R^d with e_i the ith natural basis vector in R^d. Then

$$(29)\quad \begin{aligned} \frac{\partial u^{\varepsilon}(t,x)}{\partial t} &= -\sum_{i=1}^{d} \nabla_i^{\varepsilon-}[a_i(\tfrac{x}{\varepsilon})\nabla_i^{\varepsilon+}u^{\varepsilon}(t,x)] \\ &:= L^{\varepsilon}u^{\varepsilon}(t,x) \end{aligned}$$

is the diffusion equation on the ε-spaced lattice with density $u(x)$ and conductivity $a_i(x/\varepsilon)$. Our construction of an effective parameter representation of the thermal conduction process as $\varepsilon \to 0$ will be based on an asymptotic analysis of (29) using the methods in (Bensoussan et al. 1978).

Remark: Although we shall not use probablistic methods in the analysis, the associated probabilistic problem has a great deal of intuitive appeal. The operator L^{ε} may be identified as the infinitesimal generator of a pure jump process $X^{\varepsilon}(s)$ in the "slow" time scale $s: = \varepsilon^2 t$; cf. (Breiman 1968). Moreover, L^{ε} is selfadjoint on Z^d with the inner product

$$(30) \qquad (f,g) := \sum_{x \in Z^d} (f(x)g(x)).$$

Hence, the backwards and forwards equations for the process $X^{\varepsilon}(s)$ are, respectively,

$$(31) \qquad \begin{aligned} \frac{\partial p^{\varepsilon}(y,t|x)}{\partial t} &= [L^{\varepsilon}p^{\varepsilon}(y,t|\cdot)](x) \\ \frac{\partial p^{\varepsilon}(y,t|x)}{\partial t} &= [L^{\varepsilon}p^{\varepsilon}(\cdot,t|x)](y) \end{aligned}$$

So the process is "symmetric" in the sense of Markov processes (Breiman 1968).

The asymptotic analysis of (29), when interpreted in this context, means that as the bond lattice is contracted by ε and time is sped up by ε^{-2}, the jump process $\{X^{\varepsilon}(s)\}$ approaches a diffusion process with diffusion matrix Q. In other words, on the microscopic scale thermal energy is transmitted through the lattice by a jump process; but when viewed on a macroscopic scale the energy appears to diffuse throughout the lattice. The microscopic physics are described in (Kirkpatrick 1973) and (Kittel 1976).

Because the basic problem (29) is "parabolic", we can introduce the probabilistic mechanism and make use of it in the analysis. In the "hyperbolic" problems we treat later, this device is not available.

B. Asymptotic analysis-homgenization

The essential mathematical step is to show strong convergence of the semigroup of L^{ε}, say

(32)
$$T^{\varepsilon}(t):=\exp(L^{\varepsilon}t) \underset{\varepsilon \downarrow 0}{\to} T(t):=\exp(Lt)$$

and identify the limiting operator

(33)
$$L=\sum_{i,j=1}^{d} q_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

This is accomplished by proving convergence of the resolvents

(34) for $\alpha > 0$, $[-L^{\varepsilon}+\alpha]^{-1} \underset{\varepsilon \downarrow 0}{\to} [-L+\alpha]^{-1}$

That is, if f is a given function and

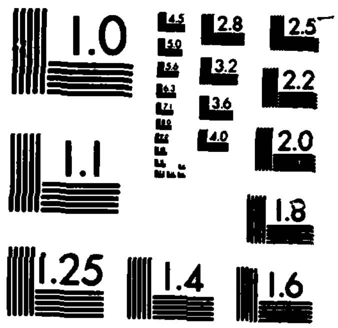
(35)
$$u^{\varepsilon}(\cdot):=[-L^{\varepsilon}+\alpha]^{-1}f$$
$$u(\cdot):=[-L+\alpha]^{-1}f$$

then $u^{\varepsilon} \to u$ (in some sense).

The method of multiple scales developed in (Bensoussan et al. 1978) will be used to prove the limit. Because the conductivities $a_i(x)$ in (29) do not depend on time, we may work directly with L^{ε} rather than the parabolic PDE (3) (cf. (Bensoussan et al. 1978 Remark 1.6, p. 242). The method of multiple scales is convenient because it is a systematic way of arriving at the "right answers" - something which is not always simple in this analysis.

AD-A150 736 MODELING AND CONTROL OF LARGE FLEXIBLE STRUCTURES(U) 2/2
SYSTEMS ENGINEERING FOR POWER INC ANNANDALE VA
B AVRAMOVIC ET AL. 31 JUL 84 SEPI-TR-84-9
UNCLASSIFIED AFOSR-TR-85-0075 F49620-83-C-0159 F/G 12/1 NL

END


1.0
4.5
5.0
5.6
6.3
7.1
8.0
2.8
3.2
3.6
4.0
2.5
2.2
2.0
1.1
1.8
1.25
1.4
1.6

Following (Bensoussan et al. 1978) and bearing in mind (35), we consider

$$(L^{\varepsilon}u^{\varepsilon})(x)=f(x) \tag{36}$$

with boundary conditions to be specified later. We shall look for u^{ε} in the form

$$u^{\varepsilon}(x)=u_0(x,\tfrac{x}{\varepsilon})+\varepsilon u_1(x,\tfrac{x}{\varepsilon})+\varepsilon^2 u_2(x,\tfrac{x}{\varepsilon})+\ldots \tag{37}$$

with the functions $u_j(x,y)$ periodic in $y \varepsilon \varepsilon Z^d$ for every $j = 0,1,\ldots$. (As it turns out the boundary conditions are somewhat irrelevant to the construction of "right answers.") To present the computations in a simple form, it is convenient to introduce $y = x/\varepsilon$, to treat x and y as independent variables, and to replace y by x/ε at the end.

Recall the operators $\nabla_i^{\varepsilon\pm}$ from (29). Applied to a smooth function $u = u(x,x/\varepsilon)$, we have

$$\begin{aligned}(\nabla_i^{\varepsilon-}u)(x,y) &= \frac{1}{\varepsilon}[u(x-\varepsilon e_i, y-e_i)-u(x,y)]\\ &= \frac{1}{\varepsilon}[u(x,y-e_i)-u(x,y)]\\ &+\frac{I}{\varepsilon}[u(x-e_i, y-e_i)-u(x,y-e_i)]\\ &-\frac{1}{\varepsilon}(\nabla_i^- u)(x,y)-\frac{\partial u}{\partial x_i}(x,y-e_i)+\varepsilon\frac{1}{2}\frac{\partial^2 u}{\partial x_i 2}(x,y-e_i)+0(\varepsilon^2)\end{aligned} \tag{38}$$

where on functions $\phi=\phi(y)$

$$(\nabla_i^- \phi)(y) = \phi(y-e_i)-\phi(y) \tag{39}$$

Defining

$$(40)\quad (\nabla_i^+ \phi)(y) = \phi(y+e_i) - \phi(y)$$

we also have

$$(41)\quad (\nabla_i^{\varepsilon+} u)(x,y) = \frac{1}{\varepsilon} (\nabla_i^+ u)(x,y) + \frac{\partial u}{\partial x_i}(x,y+e_i) + \varepsilon \frac{1}{2} \frac{\partial^2 u}{\partial x_i^2}(x,y+e_i) + 0(\varepsilon^2)$$

Now we substitute (37) into (36) and use the rules (39) (40). Equating coefficients of like powers of ε, this leads to a sequence of equations for u_0, u_1,... Specifically, (using the summation convention)

$$(42)\quad \begin{aligned}
(L^{\varepsilon} u^{\varepsilon})(x,y) &= -\nabla_i^{\varepsilon-} [a_i(y) \nabla_i^{\varepsilon+} u^{\varepsilon}] \\
&= -\frac{1}{\varepsilon^2} \nabla_i^- [a_i(y) \nabla_i^+ u_0(x,y)] \\
&\quad -\nabla_i^{\varepsilon} [a_i(y) \frac{\partial u_0}{\partial x_i}(x,y+e_i)] \\
&\quad -\frac{1}{2} \varepsilon \nabla_i^{\varepsilon-} [a_i(y) \frac{\partial^2 u_0}{\partial x_i^2}(x,y+e_i) + 0(\varepsilon) \\
&\quad -\frac{1}{\varepsilon} \nabla_i^- [a_i(y) \nabla_i^+ u_1(x,y)] \\
&\quad -\varepsilon \nabla_i^{\varepsilon-} (y) \frac{\partial u_1}{\partial x_i}(x,y+e_i)] + 0(\varepsilon) \\
&\quad -\nabla_i^- [a_i(y) \nabla_i^+ u_2(x,y)] + 0(\varepsilon) = f(x)
\end{aligned}$$

That is, labelling each term by its order in ε

$$(43)\quad \left(\varepsilon^{-2}\right) \quad \nabla_i^- [a_i(y) \nabla_i^+ u_0] = 0$$

$$(44)\quad \left(\varepsilon^{-1}\right) \varepsilon \nabla_i^{\varepsilon-} [a_i(y) \frac{\partial u_0}{\partial x_1}(x,y+e_i)] + \nabla_i^- [a_i(y) \nabla_i^+ u_1(x,y)] = 0$$

and (recall $\varepsilon \nabla_i^{\varepsilon \pm} \sim 0(1)$ in ε)

$$\frac{1}{2}\,\varepsilon\,\nabla_i^{\varepsilon-}\;[a_i(y)\,\frac{\partial^2 u_0}{\partial x_i{}^2}\;(x,y+e_i)]$$

$$(45)\quad \varepsilon^0\quad -\varepsilon\,\nabla_i^{\varepsilon-}\;[a_i(y)\,\frac{\partial u_1}{\partial x_i}\;(x,y+e_i)]$$

$$-\nabla_i^-\,[a_i(y)\;\nabla_i^+\,u_2\;(x,y)] = f(x)$$

From (43) we have

$$(46)\quad \begin{aligned} &a_i(y-e_i)[u_0(x,y) - u_0(x,y-e_i)] \\ &-a_i(y)[u_0(x,y+e_i) - u_0(x,y)] = 0 \end{aligned}$$

If we take $u_0(x,y) = u_0(x)$, this is trivially true. (We must justify this choice in subsequent steps.) And (44) simplifies to

$$(47)\quad \varepsilon\nabla_i^{\varepsilon-}[a_i(y)\,\frac{\partial u_0}{\partial x_i}\,(x)] + \nabla_i^-[a_i(y)\;\nabla_i^+\,u_1(x,y)] = 0$$

At this point we invoke a standard device in homogenization asymptotic analysis, namely, the use of "correctors." We assume

$$(48)\quad u_1(x,y) = \sum_{k=1}^{d} \chi_k(y)\,\frac{\partial u_0}{\partial x_k} + \tilde{u}_1(x)$$

with $\chi_k(\cdot)$ the correctors. Using this in (47), we have (agair using the summation convention)

$$(49)\quad \nabla_i^-\;[a_i(y)\;\nabla_i^+\;\chi_k(y)]\;\frac{\partial u_0}{\partial x_k} + [a_k(y-e_k) - a_k(y)]\;\frac{\partial u_0}{\partial x_k} = 0$$

If we take $\chi_k(y)$ as the solution of

$$(50)\qquad \nabla_i^-[a_i(y)\;\nabla_i^+\,\chi_k(y)] + [a_k\;(y-e_k)\;-a_k(y)]=0$$

(we have to verify the well-posedness of (50)), then (49) is satisfied. (The term $\tilde{u}_1(x)$ is determined (formally) from the $O(\varepsilon)$

term in the system (37) (42).)

Regarding the well-posedness of (50), note that

$$(51) \quad \nabla_i^- [a_i(y)\ \nabla_i^+ \phi(y)\ ] = \Psi\ (y)$$

has a periodic solution on εZ^d which is unique up to an additive constant if the average of the function $\Psi(y)$ over a period $(\varepsilon\ell)$ is zero; i.e.,

$$(52) \quad \bar{\Psi} := \frac{1}{\ell} \sum_{k=1}^{\ell} \Psi(y+ke_n) = 0 \quad n=1,2,\ldots,d$$

This condition clearly holds in (50), and so, $\chi_k(y)$ is well defined (up to an additive constant).

We shall determine the equation for $u_0(x)$ by using (48) (50) in (45). Using the Kronecker delta function δ_{ik}, we have

$$(53) \quad \begin{aligned} &\frac{1}{2}\varepsilon\nabla_i^{\varepsilon-}[a_i(y)\ \delta_{ik}]\frac{\partial^2 u_0}{\partial x_i \partial x_k} \\ &-\varepsilon\nabla_i^{\varepsilon-}[a_i(y)\chi_k(y+e_i)]\ \frac{\partial^2 u_0}{\partial x_i \partial x_k} = f(x) \\ &=\{\frac{1}{2}\nabla_i^-[a_i(y)\delta_{ik}] - \nabla_i^-[a_i(y)\ \nabla_i^+\chi_k]\}\ \frac{\partial^2 u_0}{\partial x_i \partial x_k} \\ &-\nabla_i^-[a_i(y)\ \chi_k(y)]\frac{\partial^2 u_0}{\partial x_i \partial x_k} - \nabla_i\ -[a_i(y)\nabla_i + u_2] = f \end{aligned}$$

The term in braces is zero from (50). To obtain the solvability condition (52) for u_2 in (53), we introduce the average

$$(54) \quad \frac{1}{2} q_{ik} = \text{symmetric part}\ [\ \{\overline{-\nabla_i^-\ [a_i(y)\ \chi_k\ (y)\ ]}\}$$

Then solvability of (53) for u_2 gives the equation

$$(55) \quad \frac{1}{2} \sum_{i,k=1}^{d} q_{ik} \frac{\partial^2 u_0}{\partial x_i \partial x_k} = f(x)$$

And this is the diffusion equation which defines the limiting behavior of the system (36) in the macroscopic x-scale in the limit as $\varepsilon \downarrow 0$.

We can justify the asymptotic analysis by using energy estimates or probabilistic methods as in (Bensoussan et al. 1978). (See also Kunnemann 1983).) We shall omit this analysis here.

C. Summary

Returning to the original problem (30) for the evolution of thermal energy on a microscopic scale, we have shown that the thermal density $u^{\varepsilon}(t,x) \to u_0(t,x)$ as $\varepsilon \downarrow 0$ (in an appropriate norm) where

$$(56) \quad \frac{\partial u_0}{\partial t} = \frac{1}{2} \sum_{i,j=1}^{d} q_{ij} \frac{\partial^2 u_0}{\partial x_i \partial x_j}$$

with

$$(57) \quad q_{ij} = -\frac{1}{\ell} \sum_{k=1}^{\ell} \{\nabla_i^- [a_i(y)\chi_k(y)] + \nabla_k^- [a_k(y)\chi_j(y)]\}$$

and the correctors χ_k, $k = 1,2,\ldots,d$, are given by

$$(58) \quad \sum_{i=1}^{d} \nabla_i^- [a_i(y) \nabla_i^+ \chi_k(y)] = -[a_k(y-c_k) - a_k(y)] \qquad k = 1,2,\ldots,d$$

To compute the limiting "homogenized" model (56), one must solve the system (58) (numerically) and then evaluate the average (57).

The fact that the original problem (30) is "parabolic" (i.e., it describes a jump random process), enables us to exploit the associated probabilistic structure to anticipate and structure the analysis. In this way we can anticipate that the limit problem will involve a diffusion process. In fact, the arguments used are entirely analytical* and the limiting diffusion (56) is constructed in a systematic way. It is not postulated.

5.3.2. Continuum Model for a Simple Structural Mechanical System

A. Problem definition

Consider the truss shown in Figure 5.2 (undergoing an exaggerated deformation)

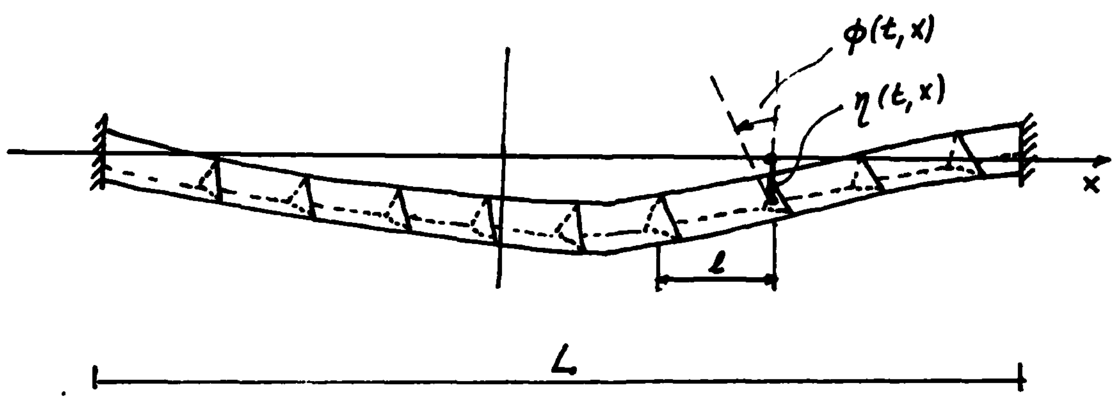


Figure 5.2. Deformed truss with triangular cross-section.

*Probabilistic arguments can be used (Bensoussan et al. 1978, Chapter 3); and they have some advantages.

We shall assume that the truss has a regular (e.g., triangular) cross-section and no "interlacing" supports. We assume that the displacements of the system are "small" in the sense that no components in the system buckle. We are interested in describing the dynamical behavior of the system when the number of cells (a unit between two (triangular) cross sections) is large; that is, in the limit as

(59) $\varepsilon := \ell/L \to 0$

We shall make several assumptions to simplify the analysis. First, we shall assume that the triangular sections are essentially rigid, and that all mobility of the system derives from the flexibility of the members connecting the triangular components. Second, we shall ignore damping and frictional effects in the system. Third, we shall confine attention to small transverse displacements $\eta(t,x)$ and small in plane rotations $\phi(t,x)$ as indicated in Figure 5.2. We shall ignore longitudinal and out of plane motions and torsional twisting. Fourth, we shall assume that the mass of the triangular cross members dominates the mass of the interconnecting links.

Systems of this type have been considered in several papers including (Noor et al. 1978) (Nayfeh and Hefzy 1978) (Anderson 1981) and (Renton 1984). In those papers a continuum beam model was hypothesized and effective values for the continuum system parameters were computed by averaging the associated parameters of the discrete system over appropriate cell volumes or areas. Our approach to the problem is based on homogenization-asymptotic analysis and is quite different from the methods used in these papers.

The assumptions made above simplify the problem substantially. By assuming the cross sectional components to be rigid and ignoring out of plane, longitudinal and torsional motions, we have effectively eliminated the geometric structure of the truss. We can retain this structure by writing dynamical equations for the nodal displacements of the truss members. For triangular cross sections nine parameters describe the displacements of each sectional element. The analysis which follows may be carried over to this case, but the algebraic complexity prevents a clear presentation of the main ideas. As suggested in (Noor et al. 1978) one would need a symbolic manipulation program like MACSYMA to carry out the complete details of the calculations. We shall take up this problem on another occasion; for now we shall treat the highly simplified problem which, as we shall see, leads to the one dimensional Timoshenko beam (and from there, under certain constraints on the parameters, to the Rayleigh and Euler beam models).

We shall begin by reformulating the system in terms of a discrete element model as suggested in (Crandal et al. 1980); see Figure 5.3.

In this model we follow the displacement $\eta_i(t)$ and rotation $\phi_i(t)$ of the ith mass M. The bending springs (k_b^i) tend to keep the system straight by keeping the masses parallel and the shearing (k_s^i) tend to keep the masses perpendicular to the connecting links. We assume small displacements and rotations so the approximations

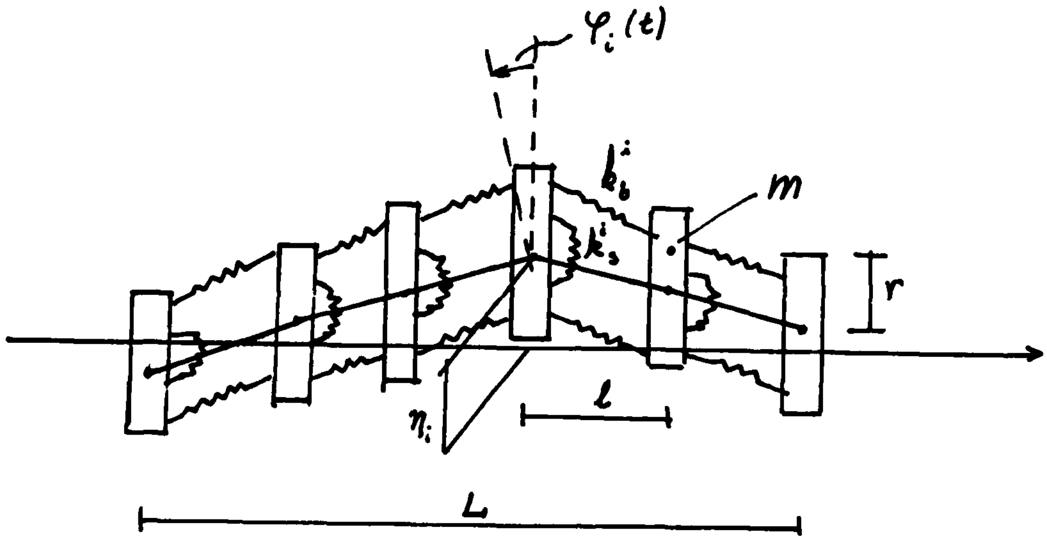


Figure 5.3 A lumped parameter model of the simplified truss system.

$$\sin\ \phi_i\ (t) \simeq \phi_i\ (t)$$

(60)

$$\tan^{-1}\ (\eta_i\ (t)\ \ell \simeq \eta_i\ (t)/\ell$$

are valid.

In this case the (approximate) equations of motion of the ith mass are

(61a)

$$\ddot{\phi}_i\ (t) = \frac{1}{r}\, k_s^i\ \{[\frac{\eta_{i+1}\ (t)\ -\ \eta_i\ (t)}{\ell}] - \phi_i\ (t)\}$$

$$\dot{s}_\ell^-\ \{K_b^i\ [\ \frac{\phi^{i+1}\ (t)\ -\phi_i\ (t)}{\ell}\ ]\ \}$$

(The spring constants depend on i since they represent the restorative forces of flexed bars, bend by different amounts.)

(61b) $\ddot{\eta}_i(t) = S_\ell^- \{ k_s{}^i [\frac{\eta_i(t)-\eta_{i+1}(t)}{\ell}] - \phi_i(t)] \}$

where we have normalized $m = 1$ and defined

(62) $S_\ell^- \eta_i := \frac{1}{\ell} [\eta_{i-1} - \eta_i]$

and similarly for $S^-\phi_i$. (We shall omit treatment of the boundary conditions at the ends of the system.)

To proceed, we shall introduce the nondimensional variable $\varepsilon = \ell/L$ and rewrite the system (61) as

(63)
$$r\ddot{\phi}_i{}^\varepsilon(t) = \frac{1}{r} K_s{}^i \{\nabla^{\varepsilon+}\eta_i{}^\varepsilon(t) - \phi_i{}^\varepsilon(t)\} + \nabla^{\varepsilon+} \{K_b{}^i \nabla^{\varepsilon-}\phi_i{}^\varepsilon(t)\}$$
$$\ddot{\eta}_i(t) = -\nabla^{\varepsilon-} \{K_s[\nabla^{\varepsilon+}\eta_i(t) - \phi_i(t)] \}$$

where

(64)
$$K_s{}^i = k_s{}^i L, \; K_b^i = k_b^i L$$
$$\nabla^{\varepsilon+}\eta_i = \frac{1}{\varepsilon} (\eta_{i+1} - \eta_i), \; \nabla^{\varepsilon-}\eta_i = \frac{1}{\varepsilon} (\eta_i - \eta_{i-1})$$

Next we associate a position in the system $x_i \in [-1/2, 1/2]$ (normalizing $L = 1$) with each mass; and we introduce the notation

(65) $\eta(t,x_i) = \eta_i(t), \; \phi(t,x_i) = \phi_i(t)$

Having normalized $L = 1$, we have $\varepsilon = \ell$ and $x_{i+1} = x_i + \ell = x_i + \varepsilon$. Let $\tilde{Z} = \{x_i\}$ be the set of all points in the system. In this notation

(66)
$$(\nabla^{\varepsilon+}\eta)(t,x) = \frac{1}{\varepsilon}\,[\eta(t,x+\varepsilon) - \eta(t,x)]$$
$$(\nabla^{\varepsilon-}\eta)(t,x) = \frac{1}{\varepsilon}\,[\eta(t,x) - \eta(t,x-\varepsilon)]\,, \quad x\varepsilon\ \tilde{Z}$$

and the system is

(67)
$$\ddot{\phi}^{\varepsilon}(t,x_i) = K_s\ (x_i)\{\nabla^{\varepsilon+}\eta^{\varepsilon}(t,x_i) - \phi^{\varepsilon}(t,x_i)\}$$
$$+r\nabla^{\varepsilon+}\{K_b(x_i)\nabla^{\varepsilon-}\phi^{\varepsilon}(t,x_i)\}$$
$$\ddot{\eta}^{\varepsilon}(t,x_i) = -\nabla^{\varepsilon-}\{K_s(x_i)[\nabla^{\varepsilon+}\eta^{\varepsilon}(t,x_i) - \phi^{\varepsilon}\ (t,x_i)]\}$$

The scaling of (67) may be interpreted in the following way: Formally, at least, the right sides of both terms in (67) are $O(\varepsilon^{-2})$. This implies that the time variations are taking place in the "fast time scale" $\tau = t/\varepsilon$. Also, the spatial variations are taking place in the microscopic scale" x which varies in ε-increments (e.g., $x_{i+1} = x_i + \varepsilon$). Introducing the macroscopic scale $z = \varepsilon x$, and the slow time scale $\sigma = \varepsilon\tilde{\tau}$, we may rescale (67) and observe its dynamical evolution on the large space-time scale on which macroscopic events (e.g., "distributed phenomena") take place.

Rewritten in this spatial scale, the system becomes

(68a)
$$\frac{d^2\phi^{\varepsilon}(t,\frac{z_i}{\varepsilon})}{dt^2} = \frac{1}{\varepsilon}\,K_s\ (\frac{z_i}{\varepsilon})\ \{\delta^{\varepsilon+}\eta^{\varepsilon}(t,\ \frac{z_i}{\varepsilon})$$
$$-\phi^{\varepsilon}(t,\ \frac{z_i}{\varepsilon}\ )\}+$$
$$+\ \frac{1}{\varepsilon^2}\ \delta^{\varepsilon+}\{rK_b\ (\frac{z_i}{\varepsilon})\,\delta^{\varepsilon-}\phi^{\varepsilon}(t,\frac{z_i}{\varepsilon})$$

(68b)
$$\frac{d^2\eta^{\varepsilon}(t,\ \frac{z_i}{\varepsilon}\ )}{dt^2} = \frac{1}{\varepsilon^2}\ \delta^{\varepsilon-}\ \{k_s\ (\ \frac{z_i}{\varepsilon}\ )\ [\,\delta^{\varepsilon+}\eta^{\varepsilon}\ \ (t,\frac{z_i}{\varepsilon} - \varepsilon\phi^{\varepsilon}\ \ (t,\frac{z_i}{\varepsilon})\,]\}$$

where

(69) $\delta^{\varepsilon\pm} = \varepsilon\nabla^{\varepsilon\pm} = O(1)$ in ε

The essential mathematical problem is to analyze the solutions ψ^{ε}, η^{ε} of (68) in the limit as $\varepsilon\downarrow 0$.

B. Mathematical analysis

To proceed, we shall generalize the problem (68) slightly by allowing K_s and K_b to depend on z as well as z/ε. This permits the restoring forces in the model system to depend on the large scale shape of the structure as well as on local deformations. We use the method of multiple scales; that is, we look for solutions of (10) in the form

$$\begin{aligned} \eta^{\varepsilon}(t) &= \eta^{\varepsilon}(t,z,y) \qquad y=z/\varepsilon \\ \phi^{\varepsilon}(t) &= \phi^{\varepsilon}(t,z,y) \end{aligned} \tag{70}$$

and we have

$$K_s = K_s\,(z,y),\ K_b = K_b(z,y),\ y = z/\varepsilon \tag{71}$$

On smooth functions $\psi(z,\frac{z}{\varepsilon})$ the operators $\delta^{\varepsilon\pm}$ satisfy

$$\begin{aligned} (\delta^{\varepsilon+}\,\psi)\ (z,y) &= \psi\ (z+\varepsilon,y+1) - \psi\,(z,y) \\ &= \psi\ (z,y+1) - \psi\,(z,y) + \psi\ (z+e,\ y+1) - \psi\ (z,y+1) \\ &= (\delta^{+}\psi)\ (z,y) + \varepsilon\ \frac{\partial\psi}{\partial z}\ (z,y+1) + \frac{1}{2}\ \frac{\varepsilon^2\partial^2\psi}{\partial z^2}\ (z,y+1) + O\,(\varepsilon^3) \end{aligned} \tag{72a}$$

$$(\delta^{\varepsilon-}\psi)\quad(z,y) = \psi(z,y) - \psi(z-\varepsilon,y-1)$$

$$(72b)\qquad = \psi(z,y) - \psi(z,y-1) + \psi(z,y-1) - \psi(z-\varepsilon,y-1)$$

$$= (S^{-}\psi)\quad(z,y) - \varepsilon\frac{\partial\psi}{\partial x}\quad(z,y-1) + \frac{1}{2}\varepsilon^{2}\frac{\partial^{2}\psi}{\partial z^{2}}\quad(Z,y-1) + 0(\varepsilon^{3})$$

We assume that ψ^{ε}, η^{ε} may be represented as

$$(73)\qquad \begin{aligned}\psi^{\varepsilon}\ (t,z,y) &= \psi_{0}\ (t,z) + \varepsilon\psi_{1}\ (t,z,y) + \dots\\ \eta^{\varepsilon}\ (t,z,y) &= \eta_{0}\ (t,z) + \varepsilon\ \eta_{1}\ (t,z,y) + \dots\end{aligned}$$

and substituting (73) in (68) and using (71) (72), we arrive at a sequence of equations for $(\psi_{\sigma}\ \eta_{1})$, (ψ_{1}, η_{1}),... by equating the coefficients of like powers of ε.

Starting with ε^{-2}, ε^{-1}, ε^{0}, ..., we have

$$(74)\quad \frac{1}{\varepsilon^{2}} S^{+}[r\ K(z,y)\ S^{-}\ \psi_{0}\ (t,z)] = 0$$

which is trivially true from (72b) (73). The same term for (68b) is trivially satisfied by the assumption (73). Continuing

$$(75)\quad \frac{1}{\varepsilon}[S^{+}\{r\ K_{b}\ (z,y)\ S^{-}\ \psi_{1}\ (t,z,y)\} + K_{s}\ (z,y)\ \{S^{+}\eta_{0}(t,z) - \psi_{0}\ (t,z)\}] = 0$$

which may be solved by using the corrector $\chi_{\phi}(z,y)$ and taking

$$(76)\quad \psi_{1}\ (t,z,y) = \chi_{\phi}\ (z,y)\ \psi_{0}\ (t,z)$$

with

$$(77)\quad S^{+}\{rK_{b}\ (z,y)\ S^{-}\chi_{\phi}\ (z,y)\} = K_{s}(z,y)$$

If we regard z as a parameter in (77), then there exists a solution

χ_ϕ, unique up to an additive constant, if $K_b(z,\cdot)$, $K_s(z,\cdot)$ are periodic in y, if there exist constants A,B so that

(78) $\quad 0 < A \leq K_b\,(z,y) \leq B < \infty$

and if the average of $K_s(z,\cdot)$ is zero

(79) $\quad \frac{1}{L} \int_{-L/2}^{L/2} K_s\,(z,y)\, dy = 0$

which holds if the system is pinned at the ends as indicated in Figure 5.3. Let us assume that (78) (79) hold, and

(80) $\quad 0 < A \leq K_s\,(z,y) \leq B < \infty$

(which we shall need shortly).

Considering (68b), the $O(\varepsilon^{-1})$ term in the asymptotic expansion is

(81) $\quad \frac{1}{\varepsilon} [\, S^- \{\, K_s\,(z,y)\,\, (S^+\eta_1\,(t,z,y)\,\,\psi_\sigma\,(t,z))\}] = 0$

Again we introduce the corrector $\chi_\eta(z,y)$, and take η_1 in the form

(82) $\quad \eta_1\,(t,z,y) = \chi_\eta\,(z,y)\,\psi_0\,(t,z)$

which gives the equation for the corrector

(83) $\quad S^- \{K_s(z,y)\,\,[S^+\chi_\eta\,(z,y)\, -1]\} = 0$

or

(84) $\quad S^-\{K_s(z,y)\,\, S^+\chi_\eta(z,y)\} = K_s(z,y) - K_s\,(z,y-1)$

By hypothesis the right side in (84) is periodic in y and has zero average (79). Hence, (84) has a periodic solution, unique to an additive constant.

Continuing, the $O(\varepsilon^0)$ term in (68a) is

$$
\begin{aligned}
&S^+ \{r\, K_b(z,y)\, S^- \psi_2\, (t,z,y)\} \\
&+K_s\, (z,y)\, [S^+\, \eta_1\, (t,z,y)\, \psi_1\, (t,z,y)] \\
&+K_s\, (z,y)\, \frac{\partial \eta^0}{\partial z}\, (t,z) \\
&+S^+ \{r\, K_b\, (z,y)\, \frac{\partial}{\partial z}\, \psi_1\, (t,z,y)\} \\
(85)\qquad &+S^+ \{r\, K_b\, (z,y)\, \frac{\partial^2}{\partial z^2}\, \psi_0\, (t,z)\} \\
&+\frac{\partial}{\partial z} \{ r\, K_b\, (z,y+1)\} \frac{\partial}{\partial z} \psi_0\, (t,z) \\
&+\frac{\partial^2}{\partial z^2} \{ r\, K_b(z,y+1)\}\, \psi_0(t,z) - \frac{\partial^2 \psi_0}{\partial t^2} = 0
\end{aligned}
$$

This should be regarded as an equation for ψ^2 as a function of y with (t,z) as parameters. In this sense the solvability condition is as before, the average of the sum of all terms on the left in (85), except the first, should be zero. We must choose ψ_0 so that this in fact occurs; and that defines the limiting system.

Using the correctors (76) (82), we must have

$$
\begin{aligned}
&\underset{(y)}{\text{Average}} \Big\{ \frac{\partial^2 \psi_0}{\partial t^2} - \frac{\partial^2 \psi_0}{\partial z^2} [\, S^+ (r\, K_b\, (z,y) + S^+\, (r\, K_b\, (s,y)\, \chi_\psi\, (z,y)\,] \\
(86)\qquad &- \frac{\partial \psi_0}{\partial z} [\, \frac{\partial}{\partial z}\, (r\, K_b\, (z,y+1))] - \frac{\partial \eta_0}{\partial z}\, K_s\, (z,y) \\
&- \psi_0\, [\frac{\partial^2}{\partial z^2}\, (r\, K_b\, (z,y+1)) + S^+\, (r\, K_b\, (z,y)\, \frac{\partial}{\partial z} \chi_\psi (z,y) \\
&+ K_s\, (z,y)\, (S^+\, \chi_\eta\, (z,y) - \chi_\psi (z,y))\Big\} = 0
\end{aligned}
$$

Defining the functions $EI(z)$, $G(z)$ by the associated averages in (86), the averaged equation is

$$(87)\quad \frac{\partial^2\psi_0}{\partial t^2} = \frac{\partial}{\partial z}\left(EI\ \frac{\partial\psi_0}{\partial z}\right) + G\frac{\partial\eta_0}{\partial z} - H\psi_0$$

which is the angular component of the Timoshenko beam system (Crandall et al. 1980 p. 348).

Arguing in a similar fashion, we can derive the equation for the displacement $\eta_0(t,z)$ in the Timoshenko beam system

$$(88)\quad \frac{\partial^2\eta_0}{\partial t^2} = \frac{\partial}{\partial z}\left[N\ (z)\ \left(\frac{\partial\eta_0}{\partial z} - \psi_0\ (t,z)\right)\right]$$

C. Summary

We have shown that a simplified model of the dynamics of the truss with rigid cross sectional area may be well approximated by the Timoshenko beam model in the limit as the number of cells $(\sim L/\ell)$ becomes large. The continuum beam model emerges naturally in the analysis, as a consequence of the periodicity and the scaling.

To compute the approximate continuum model, one must solve (77) and (84) (numerically) for the correctors and then compute the parameters in (87) (88) by numerically averaging the quantities in (86) (and its analog for (68b)) which involve the correctors and the data of the problem.

6. Homogenization and Optimal Stochastic Control

In this section and the next we show that the process of deriving effective "continuum" approximations to complex systems may be developed in the context of optimal control and state estimation designs for those systems. This procedure is more effective than the procedure of first deriving homogeneous/continuum approximations for the structure, designing a control or signal processing algorithm for the idealized model, and then adapting the algorithm to the physical model. In fact, separation of optimization and asymptotic analysis can lead to incorrect algorithms or approximations, particularly in control problems where nonlinear analysis (e.g., of the Bellman dynamic programming equation) is required. The problems treated here and in the following section are abstract systems which illustrate the basic techniques. At the beginning of section 7 we shall present a simple argument which shows how the class of models treated here might arise. In subsequent work we shall apply the combined homogenization - optimization procedure described here to the problem of controlling the dynamics of lattice structures like the truss structure analyzed in the previous section.

6.1 A Prototype Problem

The interaction of homogenization and stochastic control was discussed briefly in the book (Bensoussan, Lions and Paponicolaou 1978), and in (Bensoussan 1979) and (Blankenship 1979). The recent paper (Bensoussan, Boccardo and Murat 1984)* provides the first systematic analysis of an abstract control problem involving homogenization. We shall briefly summarize its main results against the background of the lattice system discussed in section 5.

The problem is to control x (t) given by

$$dx^{\varepsilon} = [\ g\ (x^{\varepsilon}, \frac{1}{\varepsilon} x^{\varepsilon}, v) + \frac{1}{\varepsilon} b (x^{\varepsilon}, \frac{1}{\varepsilon} x^{\varepsilon})]\ dt + \sigma (x^{\varepsilon}, \frac{1}{\varepsilon} x^{\varepsilon})\ dw\ (t)$$

(1)

$$x^{\varepsilon}(0) = x,\ 0 \leq t$$

with $x^{\varepsilon}(t)$ defined in a bounded domain $0 \subset R$ with smooth boundary. Here g, b, and σ are smooth functions of their arguments, $w(t)$ is a standard R^n - valued Wiener process, v is the control and $\varepsilon > 0$ is a parameter. We assume that g, b, and σ are periodic in their second argument with period one on the unit torus Y in R.

Let τ^{ε} be the first exit time of $x^{\varepsilon}(t)$ from the domain 0. The cost function is

$$(2)\quad J_x^{\varepsilon}\ (v(\cdot)) = E^{\varepsilon}_{v(\cdot)}[\ \int_0^{\tau\varepsilon} L(x^{\varepsilon}, \frac{1}{\varepsilon} x^{\varepsilon}, v)\ \exp\ (-\int_0^s c\ (x^{\varepsilon}, \frac{x^{\varepsilon}}{\varepsilon}, v) ds)\ dt]$$

and we define

$$(3)\quad u^{\varepsilon}(x) = \inf_{v(\cdot)} J_x^{\varepsilon}\ (v(\cdot))$$

We assume that the cost rate $L(x,y,v)$ is periodic in y on the torus, and has linear growth in v. The discount factor $c(x,y,v)$ is uniformly bounded, positive, and periodic in y. The set of admissible controls U_{ad} consists of feedback functions $v(\cdot)$

$$(4)\quad v(t) = \phi^{\varepsilon}\ (x^{\varepsilon}\ (t), \frac{1}{\varepsilon} x^{\varepsilon}\ (t), t)$$

----

* We are grateful to Professor A. Bensoussan for transmitting a preprint of this paper to us.

(We shall justify this presumed structure for the control law in more detail later.) The system has highly oscillatory coefficients; and, as in section 5, one would expect the solutions of the control problem to be well approximated by the corresponding control problem with $g(x,y,v)$, $b(x,y)$, $\sigma(x,y)$, $L(x,y,v)$, and $c(x,y,v)$ replaced by their averages (appropriately defined) over y. This is not precisely the case and one must carry through a complete asymptotic analysis to determine the exact nature of the limit and the approximation.

In (Bensoussan, Boccardo, and Murat 1984) this analysis was carried out in terms of the Hamilton-Jacobi-Bellman (HJB) equation for the optimal cost,*

(5) $$A^{\varepsilon} u^{\varepsilon} = H\left(x, \frac{1}{\varepsilon} x, u^{\varepsilon}, Du^{\varepsilon}\right), \quad u^{\varepsilon}|_{\Gamma} = 0$$

(6) $$A^{\varepsilon} = - a_{ij}\left(x,\frac{x}{\varepsilon}\right) \frac{\partial^2}{\partial x_i \partial x_j} - \frac{1}{\varepsilon} b_i\left(x,\frac{x}{\varepsilon}\right) \frac{\partial}{\partial x_i}$$

where $a_{ij} = \frac{1}{2}(\sigma\sigma^T)_{ij}$ and

(7) $$H(x,y,q,p) = \inf_{v} \{L(x,y,v) + p \cdot g(x,y,v) - q\, c(x,y,v)\}$$

Notice that the Hamiltonian H is periodic in y. The objective of the analysis is to determine the limit

(8) $$u(x) = \lim_{\varepsilon \to 0} u^{\varepsilon}(x)$$

----

* Here and in the following we use the summation convention that repeated indices ij are summed over their full range.

to identify $u(x)$ as the solution to a HJB equation, and through this to associate a "limiting stochastic optimal control problem" with the original problem. The method is "homogenization" of the nonlinear partial differential equation (5). To accomplish this, it is necessary to impose further regularity and growth conditions on the coefficients in (5)-(7).

Assume U_{ad} is a nonempty subset of a compact metric space U; and

(9) $\quad L(x,y,v) : R^n \times R^n \times U_{ad} \to R$

is continuous, periodic in y, and

(10) $\quad m_0|v|^2 - m_1 \leq L(x,y,v) \leq k\,(1+|v|^2), \; m_0>0, \; m_1 \geq 0$

Also,

(11)
$$g(x,y,v): R^n \times R^n \times U_{ad} \to R^n$$
$$\sigma(x,y,v): R^n \times R^n \times U_{ad} \to R^{n \times n}$$

are periodic in y, continuous in their other arguments, and satisfy

(12)
$$|g| \leq \bar{g}\,(1+|v|)$$
$$0 < \underline{c} \leq c \leq \bar{c}$$

Under these assumptions standard selection theorems guarantee the existence of a control law $\hat{v}(x,y,q,p)$ achieving the infimum in (7) and for any q,p fixed

$$(13)\quad |\hat{v}(v,y,q,p)| \le c(1+|p|+\sqrt{|q|})$$
$$|H(x,y,q,p)| \le c(1+|p|^2+|q|)$$

Moreover,

$$(14)\quad \begin{array}{l} H(x,y,q,0) \le c(1+|\bar{v}|^2) - q\beta, \; q \ge 0 \\ H(x,y,q,0) \ge -c_1 + qc - q\beta, \; q \le 0 \end{array}$$

for some non-negative constants c, c_1 and some arbitrary $\bar{v} \in U_{ad}$.
Also,

$$(15)\quad \begin{array}{l} H(x,y,q,p) - H(x,y,s,t) \le (p-t)\cdot q\,(x,y,\hat{v}(x,y,q,p)) \\ -(q-s)\,c(x,y,\hat{v}(x,y,q,p)) \\ \le c|\,p-t\,|\,(1+|p|+\sqrt{|q|}) + c|q-s| \end{array}$$

and a similar condition holds with the roles of (q,p) and (s,t) reversed. These growth conditions suffice for the asymptotic analysis of (5).

6.2 Invariant Measures and Correctors

Consider the second order operator

$$(16)\quad A = A_x = -a_{ij}(x,y)\frac{\partial^2}{\partial y_i \partial y_i} - b_i\,(x,y)\frac{\partial}{\partial y_i}$$

and its formal adjoint

$$(17)\quad A^* = -\frac{\partial}{\partial y_i}\left(a_{ij}\,(x,y)\frac{\partial}{\partial y_j}\right) + \frac{\partial}{\partial y_i}\left[\,b_i - \frac{\partial}{\partial y_i}a_{ij}\,\right]$$

For any x fixed let m(x,y) be the solution of

$$(18)\quad \begin{aligned} &A^* m = 0,\ m(x,y) \text{ periodic in } y \\ &m > 0,\ \int_Y m(x,y)\,dy = 1 \end{aligned}$$

with m regular (in $W^{2,p}_{loc}$, $2 \leq p < \infty$). Since x is restricted to a compact subset, we may assume

$$(19)\quad 0 < \underline{m} \leq m(x,y,) \leq \bar{m}$$

for some constants $\underline{m}$ and $\bar{m}$. Thus, m(x,y) defines a probability measure on Y for each x, which we call the invariant measure associated with A.

A key assumption in the method is that the drift term b(x,y) in (1) is "centered" in the sense

$$(20)\quad \int_Y m(x,y)\, b(x,y)\, dy = 0$$

If this assumption fails, then the asymptotic analysis takes a very different form from what follows, and the results (i.e., the limits) have a totally different character.

The centering hypothesis and the regularity assumptions mean that the "correctors" defined by

$$(21)\quad \begin{aligned} &A_x \chi^i (x,y) = -b_i (x,y),\ i=1,2,\dots,n \\ &y \to \chi^i(x,y) \text{ periodic},\ \int_Y \chi_i (x,y)\, dy = 0 \end{aligned}$$

exist and are smooth (i.e., they are C^2 functions). These functions play a key role in the homogenization procedure (Bensoussan, Lions and Papanicolaou 1978) in defining the limiting system. Solution of the system (21) is the key numerical problem in applying the homogenization technique.

6.3 Identification and Interpretation of the Limit Problem

Using the invariant measure and the correctors, we define

(22)
$$Q_{ij}(x) = \int_Y m(x,y)\ [a_{ij}(x,y) - a_{ik}\frac{\partial\chi^j}{\partial y_k} - a_{jk}\frac{\partial\chi^i}{\partial y_k} - \frac{1}{2}(b_i\chi^j + b_j\chi^i)]dy$$

$$r_j(x) = -\frac{\partial}{\partial x_i}\int_Y m(x,y)\ [a_{ij}(x,y) - a_{ik}\frac{\partial\chi^i}{\partial y_k}]dy$$

(23) $$\underline{A}\,u = -Q_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j} - r_j(x)\frac{\partial u}{\partial x_i}$$

(24) $$\underline{H}(x,q,p) = \int_Y m(x,y)\ H(x,y,q\ (I-D\chi)p)\ dy,$$
$$(D\chi_p)_j = \frac{\partial\chi^k}{\partial y_i}\ p_k$$

Using the definition (21) of the correctors, we can rewrite $Q_{ij}(x)$ as

(25) $$Q_{ij}(x) = \int_Y m(x,y)\ [a_{ij}\frac{\partial\chi^j}{\partial y_k} - a_{jk}\frac{\partial\chi^i}{\partial y_k} + a_{rk}\frac{\partial\chi^i}{\partial y_r}\frac{\partial\chi^i}{\partial y_k}]\ dy$$

which is uniformly positive definite. This with the other assumptions means that

(26) $\underline{A}\, u = \underline{H}\ (x,u,Du)\ ,\ u|_{\Gamma} = 0$

has a unique solution in $W^{2,p}$, $p \in [2,\infty)$

Equation (26) defines the "limiting control problem" associated with (1) - (4).

THEOREM (Bensoussan, Boccardo and Murat 1984) Under the assumptions of regularity and nondegeneracy of $a_{ij}(x,y)$

(27) $u^{\varepsilon} \to u$ weakly in W^{1,p_0}

for some $p_0 > 2$.

The function $\underline{H}$ in (24) may be rewritten as

$$\begin{aligned} \underline{H}\ (x,q,p) &= \inf \{ \int_Y m(x,y)\ [L\ (x,y,v(y)) \\ &+ p_i\ (g_i\ (x,y,v(y)) - \frac{\partial \chi^i}{\partial y_k}\, g_k\ (x,y,v(y)) \\ &-q\ c(x,y,v(y))]\ dy\} \qquad (28) \\ &\triangleq \inf_{v(\cdot)} \{ \tilde{L}\ (x,v(\cdot)) + p \cdot \tilde{g}\ (x,v(\cdot)) \\ &\qquad -q\ \tilde{c}\ (x,v(\cdot))\} \end{aligned}$$

where $v(\cdot)$ is any Borel function on Y with values in U_{ad}. From the final expression it is clear that H is the Hamiltonian of a control problem.

Using (22) (23) and (28), we can identify this problem explicitly as

$$u(x) = \inf_{v(\cdot)} J_x \, (v\,(\cdot)) \tag{29a}$$

$$J_x \, (v(\cdot)) = E_{v(\cdot)} \, [\int_0^{\tau} \tilde{L} \, (x(t), \, v(t)) \cdot \exp \, (-\int_0^t \tilde{c} \, (x,(s), \, v(s)) \, ds) \, dt] \tag{29b}$$

$$\begin{aligned} dx &= [\tilde{g} \, (x\,(t), \, v\,(t) \, + r \, (x\,(t))]dt + \sqrt{2Q\,(x\,(t))} \, dw\,(t) \\ x\,(0) &= x, \; 0 \leq t \end{aligned} \tag{29c}$$

with τ_x the first exit time of the (controlled) process $x(t)$ from the compact domain 0.

A key property of the limiting control problem is that the admissible control laws depend on y, the "rapidly varying state variable", as may be seen from (28). Thus, the "fine structure" of the original problem (1) - (4), that is, the periodic dependence of controls on $y^{\varepsilon}(t) = x^{\varepsilon}(t)/\varepsilon$, is retained in the limiting problem. In effect, the limiting optimal control law depends on the fine structure - property which may not be desireable in some engineering implementations.

Notice that the limiting state dynamics (29c) emerge naturally from the asymptotic analysis (see Bensoussan, Boccardo and Murat 1984 for details) of the nonlinear HJB system (5) - (7). Note further that simply averaging the functions g, b, σ, L, c with respect to y, and then posing an optimal control problem in terms of the averaged (y independent) functions leads to wrong answers for two reasons. First, the appropriate averaging process involves the invariant measure and the correctors, and the role of the latter is not obvious in a naive application of averaging methods. Second, as (29) shows, the optimal control law that emerges in the limiting process depends on y, which cannot be the case when the averaging process is separated from the optimization process.

The key numerical problems in applying the homogenization - optimization procedure to a specific problem, e.g., control of the lattice structure described in section 5, are

1. solving for the invariant measure (18),

2. verifying the centering hypothesis (20), solving for the correctors (21),

3. computing the averaged quantities (22) - (24), and

4. solving the limiting control problem (26) - (29).

Sequential solutions of these problems constitute algorithms for simultaneous homogenization and control.

6.4 Application: Homogenization - optimization of lattice structures

To illustrate the techniques of the last sections, we shall reconsider the model for the lattice structure analyzed in subsection 5.3.2 with control actuators added. The truss shown in Figure 5.3 is again constrained to move in the plane and torsional motion is excluded to simplify the model and confine attention to the basic ideas. Now, however, we include a finite number of actuators acting to cause transverse motions. The truss with actuator forces indicated by arrows is shown in Figure 6.1. The corresponding discrete element model is shown in Figure 6.2.

Suppose that the physical actuators act along the local normal to the truss midline as shown in the figures, and that the forces are small so that linear approximations to transcendental functions (e.g., $\sin \psi_i \approx \psi_i$, etc.) are valid. Then the controlled equations of motion of the discrete element system are (recall equation (5.63))

$$
(30)\quad
\begin{aligned}
r \ddot{\psi}_i^{\varepsilon}(t) &= \frac{1}{r} K_s^i [\nabla^{\varepsilon+} \eta_i^{\varepsilon}(t) - \psi_i^{\varepsilon}(t)] + \nabla^{\varepsilon+}[K_b^i \nabla^{\varepsilon-} \phi_i^{\varepsilon}(t)] \\
\ddot{\eta}_i^{\varepsilon}(t) &= -\nabla^{\varepsilon-}\{K_s^i [\nabla^{\varepsilon-} \eta_i^{\varepsilon}(t) - \dot{\psi}_i^{\varepsilon}(t)]\} + \sum_{j=1}^{m} \delta(i, i_j)\, u_j(t)
\end{aligned}
$$

where the notation in (5.64) has been used,

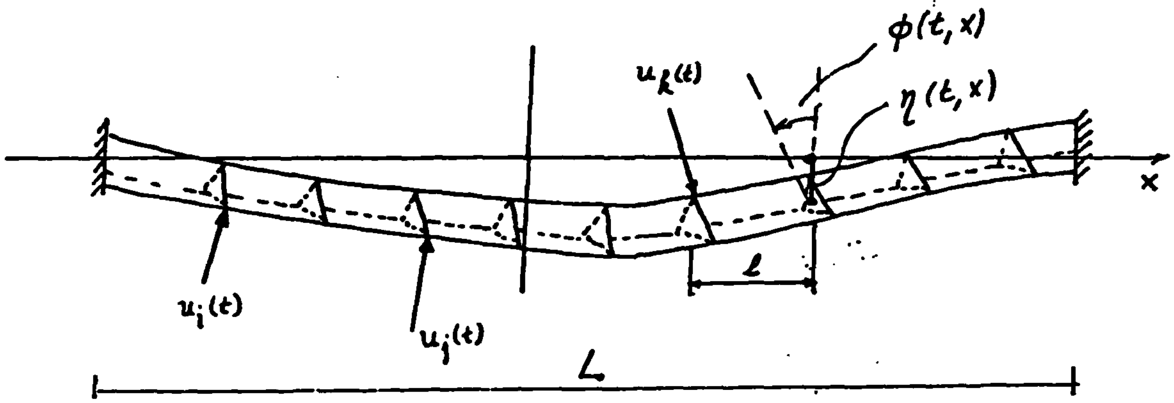


Figure 6.1 Truss with transverse actuator forces.

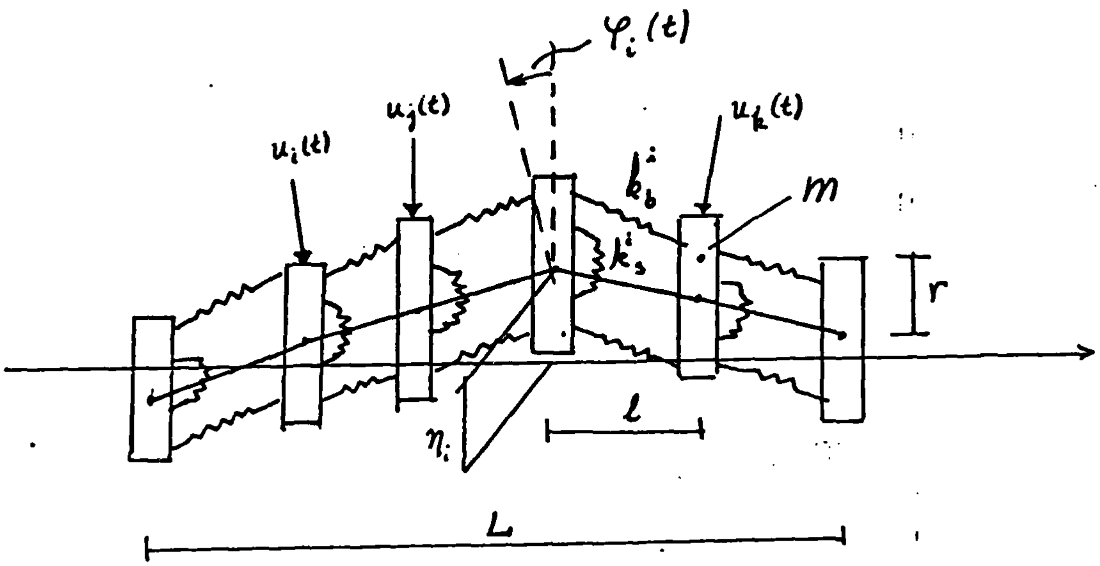


Figure 6.2 Discrete element model of the controlled truss.

$$\delta(i,j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{31}$$

and i_j, $j=1,\ldots,m$ are the locations of the actuators. Hence, if $\delta(i,i_j) = 0$ for all $j=1,\ldots,m$ there is no actuator located at the ith point which correponds to the physical point $x_i \varepsilon [0,L]$. The number m of actuators is given at the outset and does not, of course, vary with the scaling.

The control problem is to select the actuator forces as functions of the displacements and velocities of components of the structure to damp out motions of the structure. Measurements would typically be available from a finite number of sensors located along the structure. We shall not elaborate on this component of the model, and shall instead assume that the entire state can be measured. To achieve the stabilization, we shall associate a cost functional with the system (30). Let

$$u(t) = [u_1(t), \ldots, u_m(t)] \tag{32}$$

be the vector of control forces, and

$$\begin{aligned} J^\gamma [u(\cdot)] = \int_0^\infty \sum_{i=1}^N \{ & a_i [\psi_i^\varepsilon(t)]^2 + b_i [\eta_i^\varepsilon(t)]^2 + \alpha_i [\dot{\psi}_i^\varepsilon(t)]^2 \\ & + \beta_i [\eta_i^\varepsilon(t)]^2 + \sum_{j=1}^m \delta(i,i_j)\, u_j^2(t) \} e^{-\gamma t}\, dt \end{aligned} \tag{33}$$

where (a_i, b_i) and (α_i, β_i) are non-negative weignts. Formally, the control problem is to select $\delta(i,i_j)u_j(t)$, $i=1,\ldots,N$, $j=1,\ldots,m$ to achieve

$$\inf_{u(\cdot)} J^\gamma [u(\cdot)] \tag{34}$$

subject to (30) (31) and the appropriate boundary conditions. The case $\gamma \to 0$ corresponds to stabilization by feedback.

The analysis of this control problem is based on the scaling used in section 5, equations (5.63) - (5.69). Let $\tau = t/\varepsilon$ be the fast time scale, then

$$J^{\gamma}[u(\cdot)] = \int_0^{\infty} \varepsilon \sum_{i=1}^{N} \{a_i[\tilde{\psi}_i^{\varepsilon}(\tilde{\tau})]^2 + b_i[\tilde{\eta}_i^{\varepsilon}(\tilde{\tau})]^2 \tag{35}$$

$$+ \alpha_i \varepsilon^2 [\dot{\tilde{\psi}}_i^{\varepsilon}(\tau)^2 + \beta_i \varepsilon^2 [\dot{\tilde{\eta}}_i^{\varepsilon}(t)]^2 + \sum_{j=1}^{m} \delta(i,i_j)[u_j(\tilde{\tau})]^2\} e^{-\varepsilon\gamma\tau} d\tau$$

with $\tilde{\psi}_i^{\varepsilon}(\tau) = \psi_i^{\varepsilon}(\varepsilon\tau)$, etc.

Let $(\psi, \dot{\psi}, \eta, \dot{\eta})$ be the state vector of the system (30) with $\psi = [\psi_1, \ldots, \psi_N]$ and similarly for the other terms. Let $V = V^{\varepsilon,\gamma}(\psi, \dot{\psi}, \eta, \dot{\eta})$ be the optimal value function for the problem (30) (35). Then the Bellman problem associated with (30) (35) is

$$\varepsilon \sum_{i=1}^{N} [\dot{\psi}_i V_{\psi_i} + \dot{\eta}_i V_{\eta_i}]$$

$$+ \varepsilon \sum_{i=1}^{N} \{\frac{1}{r^2} K_s^i [\nabla^{\varepsilon+} \eta_i - \psi_i] + \frac{1}{r} \nabla^{\varepsilon+}[K_b^i \nabla^{\varepsilon-} \psi_i]\} V_{\dot{\psi}_i} \tag{36}$$

$$+ \varepsilon \sum_{i=1}^{N} [-\nabla^{\varepsilon-}[K_s^i(\nabla^{\varepsilon-}\eta_i - \psi_i)]] V_{\dot{\eta}_i} +$$

$$\min_{u_j \in u_{ad}^j} \{\varepsilon \sum_{i=1}^{N} \sum_{j=1}^{M} [\delta(i,i_j) u_j V_{\dot{\eta}_i} + \delta(i,i_j) u_j^2]\}$$

$$+ \varepsilon \sum_{i=1}^{N} [a_i \psi_i^2 + b_i \eta_i^2 + \varepsilon^2(\alpha_i \dot{\psi}_i^2 + \beta_i \dot{\eta}_i^2)] - \varepsilon\gamma V = 0$$

REMARKS:

1. Note that the minimization in (36) is well defined if the admissible range of the control forces is convex since the performance measure has been assumed to be quadratic in the control variables $\delta(i,i_j)u_j$.

2. Since we have not included the effects of noise in the model, the state equations are deterministic and the Bellman equation (36) is a first order system. To "regularize" the analysis, at least along the lines followed in conventional homogenization analysis, it is useful to include the effects of noise in the model and exploit the resulting coercivity properites in the asymptotic analysis.

3. If we introduce the macroscopic spatial scale $z = \varepsilon x$, the mesh $\{x_i\}$, and the variables

$$(37)\quad \psi\ (t,z_i) = \psi_i^{\varepsilon}\ (t),\ \dot{\psi}\ (t,z_i) = \dot{\psi}_i^{\varepsilon}\ (t), \text{ etc.}$$

then the sums $\varepsilon \sum_i$ may be regarded as Riemann approximations to integrals over the macroscopic spatial scale z. The asymptotic analysis of (36) with this interpretation defines the mathematical problem constituting simultaneous homogenization - optimization for this case.

We shall return to this challenging problem in subsequent work.

7. Homogenization and State Estimation in Heterogeneous Structures

7.1 Problem Statement and Background

Signal processing problems arise in the control of large space structures in several ways. Our special interest here is in the treatment of a nonlinear filtering problem for a prototype abstract system with a homogeneous infrastructure. We shall give a* detailed treatment of the filtering problem for the system

$$dx^{\varepsilon}(t) = g[\frac{x^{\varepsilon}(t)}{\varepsilon}]dt + \sigma[\frac{x^{\varepsilon}(t)}{\varepsilon}]dw(t)$$

$$dz^{\varepsilon}(t) = h[\frac{x^{\varepsilon}(t)}{\varepsilon}]x^{\varepsilon}(t)dt + dv(t) \tag{1}$$

$$x^{\varepsilon}(0) = \xi\ ,\quad z^{\varepsilon}(0) = 0,\ 0 \le t \le T\ ,\ \varepsilon > 0$$

where ξ is an R^n - valued random variable, g, σ, and h are periodic on the (unit) torus in R , and w(t) and v(t) are independent, standard vector-valued Wiener processes which are independent of ξ. The filtering problem for (1) is to estimate $x^{\varepsilon}(t)$, i.e., compute its conditional density, given $Z_t^{\varepsilon} = \sigma\{z^{\varepsilon}(s),\ 0 \le s \le t\}$, the σ-algebra of observations. We are interested in the behavior of this filtering problem in the limit as $\varepsilon \to 0$.

In the model (1) the vector $x^{\varepsilon}(t)$ may be regarded as the composite state of the overall system formed from the lexicographical

---

* The results in this section are joint work with A. Bensoussan at INRIA in Versailles. This research was also supported in part by the Department of Energy.

listing of the states of each of the components of the system. The periodicity of g, σ, and h represents a regularity property of the array; the small parameter ε represents a natural, non-dimensional "distance" or "coupling" variable characterizing component interactions. In a subsequent paragraph we shall describe a prototype system in this class.

One would expect the system (1) to be well approximated as $\varepsilon \to 0$ by a similar system with $g(x/\varepsilon)$, $\sigma(x/\varepsilon)$, and $h(x/\varepsilon)$ replaced by their averages $\bar{g}$, $\bar{\tau}$, and $\bar{h}$ over the torus. This is the case, although the precise nature of the average is difficult to guess from a cursory inspection of (1). The filtering problem for the associated limiting system is just the Kalman-Bucy filtering problem which has a simple, closed form solution. By constructing an asymptotic expansion for the conditional density of $x^{\varepsilon}(t)$ given Z_t^{ε}, we can obtain a family of finite dimensional linear filters which (presumably) provide increasingly accurate, e.g., $O(\varepsilon)$, $O(\varepsilon^2)$,...,etc., approximations of the conditional density of $x^{\varepsilon}(t)$. The technique used to derive the result is "homogenization" of a linear stochastic partial differential equation for the (unnormalized) conditional density of $x^{\varepsilon}(t)$ given Z.

While the system (1) is obviously only an example of a larger class of problems, we shall see that its analysis has all the essential difficulties of more general problems. Before starting the analysis it is useful to illustrate how a problem like (1) might arise.

Consider the prototype system:

$$dx_i(t) = a[x_i(t), u(t)]dt + \frac{1}{N} \sum_{j=1}^{N} b[x_j(t)]dw_{ij}(t) \; i = 1,2,\ldots, N, \quad t \geq 0$$

a and b are smooth functions of their (vector-valued) arguments,
and $w_{k\ell}$ are standard (vector) Wiener processes which are
endent for $(i,j) \neq (k,\ell)$ and $u(t)$ is a vector of control
bles. The functions a and b are the same for all the subsystems
the overall system with state $x(t) = [x_1(t),\ldots,x_n(t)]^T$ has a
eneous structure. The coupling is random and normalized by 1/N
flect the assumption that each subsystem has O(1) coupling to the
nder of the system (as opposed to O(N), O(1/N), etc.), no matter
arge the latter is.

Associated with (2), we define

$$S(t) = \sum_{i=1}^{N} x_i(t) = \text{"the aggregate output"}$$

$$\sigma(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(t) = \text{"the average output"}$$

se that in the process of controlling the system, we observe not
but the aggregate $S(t)$ through the measurement

$$dz(t) = h[S(t)]dt + dv(t)$$

h smooth and $v(t)$ a standard Wiener process. Suppose further
the control $u(t)$ is defined by $u(t) = f[\hat{S}(t)]$ with $\hat{S}(t)$ an
mate of $S(t)$ derived from $z(s)$, $s \leq t$. We would like to analyze
(4) in the limit as $N \to \infty$; and, more precisely, to show that this

analysis involves the asymptotic analysis of systems scaled like (1).

Defining $\delta x_i(t) = x_i(t) - \sigma(t)$, we have $\sum_{i=1}^{N} \delta x_i(t) = 0$. The aggregate output $S(t)$ satisfies

$$
\begin{aligned}
dS(t) &= \sum_{i=1}^{N} a(x_i(t), u(t))dt \\
(5) \qquad &+ \frac{1}{N} \sum_{j=1}^{N} b(x_j(t)) \sum_{i=1}^{N} dw_{ij}(t) \\
&= Na(\sigma(t), u(t))dt \\
&+ 0(\delta |x_i(t)|^2)dt + b(\sigma(t))\frac{1}{N} \sum_{i,j=1}^{N} dw_{ij}(t) \\
&+ b_x(\sigma(t))\frac{1}{N} \sum_{i=1}^{N} \delta x_i(t) \sum_{j=1}^{N} dw_{ij}(t) \\
&+ 0(|\delta x_i(t)|^2)d\tilde{w}(t)
\end{aligned}
$$

where $\tilde{w}(t)$ is a vector Wiener process defined from the components of $w_{ij}(t)$. Neglecting $O(|\delta x_i(t)|^2)$ terms, we have

$$
\begin{aligned}
d\sigma(t) &= a(\sigma(t), u(t))dt + b(\sigma(t))\frac{1}{N^2} \sum_{i,j=1}^{N} dw_{ij}(t) \\
(6) \qquad &+ b_x(\sigma(t))\frac{1}{N} \sum_{i=1}^{N} \delta x_i(t)\frac{1}{N} \sum_{j=1}^{N} dw_{ij}(t)
\end{aligned}
$$

To treat the last term, we use the formal argument in (Geman 1982) which goes as follows: As $N \to \infty$ a "local chaos" condition prevails in which each subsystem with state $\delta x_i(t)$ behaves "independently" of every other subsystem, and, in effect, of the noises $(\sum_{i=1}^{N} dw_{ij}(t)/N)$, $i=1,\ldots,N$. That is, a law of large numbers applies to the last term as $N \to \infty$. Since

$$\sum_{i=1}^{N} \delta x_i(t) = 0$$

by the definition of $\sigma(t)$, the last term in (6) is zero. (In a more general situation, this term would approach zero as $N \to \infty$.) Notice that

$$\frac{1}{N^2} \sum_{i,j=1}^{N} w_{ij}(t)$$

in the second term behaves like a standard Wiener process for each N. Thus, for large N we obtain the approximate model

(7) $$d\sigma(t) = a(\sigma(t), u(t))dt + b(\sigma(t))d\tilde{w}(t)$$

Now let $\hat{a}(\sigma,S) = a(\sigma,\hat{S})$ and assume that $\hat{S}$ and S have the same order behavior in N for N large. Defining $\varepsilon = 1/N$, we have two descriptions of the aggregate behavior of (2) for N large

(8a)
$$d\sigma(t) = \hat{a}(\sigma(t), \frac{1}{\varepsilon}\sigma(t))dt + b(\sigma(t))d\tilde{w}(t)$$
$$dz(t) = h(\frac{1}{\varepsilon}\sigma(t))dt + dv(t)$$

(8b)
$$dS(t) = \frac{1}{\varepsilon}\hat{a}(\varepsilon S(t), S(t))dt + \frac{1}{\varepsilon}b(\varepsilon S(t))d\tilde{w}(t)$$
$$dz(t) = h(S(t))dt + dv(t)$$

So to analyze the aggregate behavior of the original system (2) as $N \to \infty$, we can study (8a) or (8b) as $\varepsilon \to 0$. If $\hat{a}$, b, and h have a periodic or randomly recurrent dependence on their arguments, then the analysis of (8a,b) involves a homogenization problem.

The literature in mathematical physics and engineering contains many examples of systems scaled like (8) which can be effectively treated using homogenization theory (Keller 1977) (Larsen 1975, 1976). Homogenization methods have not been developed in control theory, other than the brief treatments in (Blankenship 1979) (Bensoussan 1979).

7.2 Preliminary Analysis

Let (Ω, F, P) be a probability space on which are defined two independent Wiener processes $\tilde{w}(t)$ and $z(t)$ with values in R^n and R^d, respectively. Let ξ be a Gaussian random variable with values in R^n which has mean x_0 and covariance P_0. Suppose ξ is independent of $\tilde{w}(t)$ and $z(t)$. Let F^t, $t \geq 0$, be a family of σ-algebras with $F^\infty = F$, such that $\tilde{w}(t)$ and $z(t)$ are adapted to F^t and ξ is F^0 - measureable. Let $Z = \sigma\{z(s), s \leq t\}$. Let Y be the unit torus in R^n and

$$g(y) \in L(R^n; R^n)$$

(9) $$\sigma(y) \in L(R^n; R^n) \quad ; \text{ invertible}$$

$$h(y) \in L(R^n; R^d)$$

which are defined on the torus Y, and which are sufficiently smooth there.

Let $x^\varepsilon(t)$ be the solution of the Itô equation

$$
(10) \quad \begin{aligned} dx^\varepsilon(t) &= \sigma\left[\frac{x^\varepsilon(t)}{\varepsilon}\right] d\tilde{w}(t) \\ x^\varepsilon(0) &= \xi \quad , \quad 0 \le t \le T \end{aligned}
$$

and note that $x^\varepsilon(t)$ is independent of $z(t)$. Consider the processes

$$
(11) \quad \begin{aligned} w^\varepsilon(t) &= -\int_0^t \sigma^{-1} g\left(\frac{x^\varepsilon}{\varepsilon}\right) x^\varepsilon ds + \tilde{w}(t) \\ v^\varepsilon(t) &= -\int_0^t h\left(\frac{x^\varepsilon}{\varepsilon}\right) x^\varepsilon ds + z(t) \end{aligned}
$$

and

$$
(12) \quad \begin{aligned} \mu^\varepsilon(t) = \exp\{ &\int_0^t h\left(\frac{x^\varepsilon}{\varepsilon}\right) x^\varepsilon \cdot dz + \int_0^t \sigma^{-1} g\left(\frac{x^\varepsilon}{\varepsilon}\right) x^\varepsilon \cdot d\tilde{w} \\ &- \tfrac{1}{2} \int_0^t \left| h\left(\frac{x^\varepsilon}{\varepsilon}\right) x^\varepsilon \right|^2 ds - \tfrac{1}{2} \int_0^t \left| \sigma^{-1} g\left(\frac{x^\varepsilon}{\varepsilon}\right) x^\varepsilon \right|^2 ds \end{aligned}
$$

For any finite T, one has

$$
(13) \quad E\, \mu^\varepsilon(T) < \infty
$$

which is a consequence of the following condition (see A. Bensoussan, J. L. Lions 1978)

$$
(14) \quad E \exp \delta |x^\varepsilon(t)|^2 \le c, \ \forall\, t \in [o,T]
$$

To check (14), consider the backward Cauchy problem $(a = \frac{1}{2}\sigma\sigma^*$, and we shall use the summation convention from here on)

$$\frac{\partial u}{\partial s} + a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \quad , \quad s \leq t \tag{15}$$

$$u(x,t) = \exp\left(\delta |x|^2\right)$$

Then

$$E \exp\left(\delta |x^\varepsilon(t)|^2\right) = E\, u(\xi,0). \tag{16}$$

Consider the function

$$\zeta(x,s) = \exp[P(s)|x|^2 + \rho(s)], \; P(s) \geq 0 \tag{17}$$

$$P(t) = \delta \; , \quad \rho(t) = 0$$

We have

$$\frac{\partial \zeta}{\partial s} + a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial^2 \zeta}{\partial x_i \partial x_j} = \zeta[\, \dot{\rho} + \dot{P}|x|^2 + 2\mathrm{tr}\; a\; P + 4|ax|^2 P^2]$$

$$\leq \zeta[\dot{\rho} + (\dot{P} + 4||a||P^2) + 2|\mathrm{tr}\; a|P] \tag{18}$$

$$\leq \zeta[\dot{\rho} + (\dot{P} + 4||a||P^2) + 2n\; ||a||P]$$

Choosing P and ρ so that

$$\dot{P} + 4||a||P^2 = 0, \quad \dot{\rho} + 2n\; ||a||P = 0 \tag{19}$$

we have

$$P(s) = \delta/[1 - 4||a||\delta(t-s)]$$

$$\exp(s) = 1/[1 - 4||a||\delta(t-s)]^{n/2} \tag{20}$$

By the maximum principle, $\xi(x,s) \geq u(x,s)$. Hence,

$$E \exp \delta |x^{\varepsilon}(t)| \leq E \left[\frac{\exp(\delta|\zeta|^2/[1-4\,\|a\|\,\delta t])}{(1-4\,\|a\|\,\delta t)^{n/2}} \right]$$

$$(21) \qquad = \frac{\exp\delta[(1-4\|a\|\delta t)I - 2P_0]^{-1}|x_0|^2}{\sqrt{|(1-4\|a\|\delta t)I - 2\delta P_0|}}$$

Therefore, sufficient conditions for (14) to hold are

$$(22) \qquad \begin{aligned} &1 - 4\,\|a\|\,\delta T > 0 \\ &(1 - 4\,\|a\|\,\delta T)\, I > 2\delta P_0. \end{aligned}$$

which hold if δ is sufficiently small. These conditions are independent of ε.

Because of (15) we can consider the change of probability given by the Girsanov transformation

$$(23) \qquad \left.\frac{dP^{\varepsilon}}{dP}\right|_{F^T} = \mu^{\varepsilon}(T)$$

Under the probability P^{ε} the processes $w^{\varepsilon}(t)$ and $v^{\varepsilon}(t)$ are independent standard Wiener processes. Since $w^{\varepsilon}(t)$ and $v^{\varepsilon}(t)$ are independent of F^0 under P^{ε}, ξ is independent of $w^{\varepsilon}(t)$ and $v^{\varepsilon}(t)$. Further, since $\{\mu^{\varepsilon}(t), F^t\}$ is a martingale, ξ has the same distribution under P^{ε} as under P.

7.3. The Filtering Problem

In the space $(\Omega, F, P^{\varepsilon}, F^t)$ we can write

$$
\begin{aligned}
dx^\varepsilon &= g(\tfrac{x^\varepsilon}{\varepsilon})\, x^\varepsilon dt + \sigma(\tfrac{x^\varepsilon}{\varepsilon}) dw^\varepsilon \\
x^\varepsilon(0) &= \xi \\
dz^\varepsilon &= h(\tfrac{x^\varepsilon}{\varepsilon})\, x^\varepsilon dt + dv^\varepsilon(t)
\end{aligned} \tag{24}
$$

where w^ε and v^ε are standard F^t - Wiener processes which are mutually independent. Moreover, ξ is a F^0 - Gaussian random variable with mean x_0 and covariance matrix P_0. The filtering problem associated with (24) consists in computing

$$
\pi^\varepsilon(t)(\psi) = E^\varepsilon[\psi(x^\varepsilon(t)) \mid Z^t] \tag{25}
$$

for Ψ any Borel bounded test function on R^n. It is easy to check that

$$
\begin{aligned}
\pi^\varepsilon(t)(\psi) &= \frac{E[\psi(x^\varepsilon(t))\mu^\varepsilon(t) \mid Z^t]}{E[\mu^\varepsilon(t) \mid Z^t]} \\
&= \frac{p^\varepsilon(t)(\psi)}{p^\varepsilon(t)(1)}
\end{aligned} \tag{26}
$$

where

$$
p^\varepsilon(t)(\psi) = E[\psi(x^\varepsilon(t))\mu^\varepsilon(t) \mid Z^t] \tag{27}
$$

Our purpose here is to study the behavior of this quantity as $\varepsilon \to 0$.

In subsequent arguments it is useful to have the bound

$$
E\, \mu^\varepsilon(T)^2 < C. \tag{28}
$$

To ensure this, we proceed as follows: For $s > 1$ we write

$$
\begin{aligned}
\mu^\varepsilon(T)^2 = {} & \exp\{2 \textstyle\int_0^T (hx^\varepsilon \cdot dz + (\sigma^{-1}g)x^\varepsilon \cdot d\tilde{w} \\
& -2s \textstyle\int_0^T |(h+\sigma^{-1}g)x^\varepsilon|^2 dt\} \cdot \exp\{(2s-1) \textstyle\int_0^T |(h+\sigma^{-1}g)x^\varepsilon|^2 dt\}
\end{aligned} \tag{29}
$$

From this we have

$$(30)\ E\,\mu^{\varepsilon}(T)^2 \le \left(E \exp\left\{\frac{s(2s-1)}{s-1} \int_0^T |(h+\sigma^{-1}g)x^{\varepsilon}|^2 dt\right\}\right)^{\frac{s-1}{s}}$$

Note that $s(2s-1)/(s-1)$ has a minimum on $[1,\infty)$ at some $s_0 > 1$. Thus, it suffices to check that

$$(31)\ E \exp\left\{\frac{s_0(2s_0-1)}{s_0-1} T|(h+\sigma^{-1}g)x^{\varepsilon}(t)|^2\right\} < \infty,\ t\in[0,T].$$

This is similar to (14) except that the parameter δ is fixed. Taking

$$(32)\quad \delta \triangleq \frac{s_0(2s_0-1)}{s_0-1} T \,\| h+\sigma^{-1}g \|^2$$

we require (22) which reads

$$(33)\quad \begin{aligned} &1 > 4\,\|a\|\, T^2 \frac{s_0(2s_0-1)}{s_0-1} \|h+\sigma^{-1}g\|^2 \triangleq \delta_0 \\ &(1-\delta_0)I > 2\delta P_0. \end{aligned}$$

These conditions restrict the size of T, and the extent to which they are necessary is not clear.

7.4. A Duality Form and an Expression for the Conditional Density

By introducing a certain duality formula it is possible to obtain an expression for the conditional density which is convenient for the homogenization and convergence analysis.

Let β be a deterministic function in $L^{\infty}(0,T;R^{d})$ and

$$(34)\quad \rho(t) = \exp\left\{ \int_0^t \beta \cdot dz - \tfrac{1}{2} \int_0^t |\beta|^2 ds \right\}$$

It is known that $\forall T$, the set of random variables, $\{\rho(T)\}$, obtained by varying β in $L^{\infty}(0,T;R^{d})$ is dense in $L(\Omega, Z^{T}, P; R^{d})$.

Let Ψ be a smooth, bounded function on R^{n} and let $\beta(t)$ be a smooth, bounded deterministic function on $[0,T]$ with values in R^{d}. We introduce the deterministic function $v^{\varepsilon}(x,t)$ which is the solution of

$$(35)\quad \begin{aligned} &\frac{\partial v^{\varepsilon}}{\partial t} + a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial^2 v^{\varepsilon}}{\partial x_i \partial x_j} + g_{ij}\left(\frac{x}{\varepsilon}\right) x_j \frac{\partial v^{\varepsilon}}{\partial x_i} \\ &\qquad + v^{\varepsilon} h_{ij}\left(\frac{x}{\varepsilon}\right) x_j \beta_i(t) = 0 \\ &v^{\varepsilon}(x,T) = \psi(x), \qquad T \geq t \geq 0 \end{aligned}$$

Because the coefficients are smooth, (35) has a solution in $C^{2,1}(R^{n} \times [0,T])$. Moreover, it satisfies the growth conditions

$$(36)\quad \begin{aligned} |v^{\varepsilon}(x,t)| &\leq C_{\delta} \exp \delta |x|^2 \\ |Dv^{\varepsilon}(x,t)| &\leq C_{\delta,\varepsilon} \exp 2\delta |x|^2 \end{aligned}$$

where $\delta > 0$ can be chosen arbitrarily small. Note that the first constant C_{δ} in (36) can be chosen independent of ε, but not δ.

One way to verify (36) is to use a probabilistic formula for $v^{\varepsilon}(x,t)$. Consider the equation

$$dx^\varepsilon = g\left(\frac{x^\varepsilon}{\varepsilon}\right) x^\varepsilon dt + \sigma\left(\frac{x^\varepsilon}{\varepsilon}\right) db$$

(37)

$$x^\varepsilon(t) = x$$

on a probability space (not necessarily the original one) where $b(s)$ is a standard Wiener process. Then

(38) $$v^\varepsilon(x,t) = E\,\{\psi(x^\varepsilon(T)) \exp \int_t^T h\left(\frac{x^\varepsilon}{\varepsilon}\right) x^\varepsilon \beta ds\}.$$

Therefore,

$$|v^\varepsilon(x,t)| \leq K \int_t^T E \exp C|x^\varepsilon(s)|\, ds$$

(39)

$$\leq K_\delta \int_t^T E \exp \delta|x^\varepsilon(s)|^2\, ds$$

where $\delta > 0$ may be chosen arbitrarily small. A calculation similar to (18) shows that

(40) $$E \exp \delta\, |x^\varepsilon(t)|^2 \leq k_\delta^t(0) \exp P_\delta^t(0)|x|^2$$

where

$$P_\delta^t(t) = \delta, \quad t \geq s$$

$$\dot{P}_\delta^t + 4(P_\delta^t)^2 \|a\| + 2\|g\|\, P_\delta^t = 0$$

(41) $$\frac{\dot{k}_\delta^t}{k_\delta^t} + 2P_\delta^t\, n\, \|a\| = 0\ , \quad k_\delta^t(t) = 1$$

Now

$$P_\delta^t(s) = \frac{\delta}{\exp[-2\|g\|(t-s) - 4\|a\|\delta(t-s)] \quad \dfrac{(1-\exp[-2\|g\|(t-s)])}{2\|g\|(t-s)}}$$

$$(42)\quad \frac{2\|g\|(t-s)\exp[-2\|g\|(t-s)]}{1-\exp[-2\|g\|(t-s)]} > \frac{2\|g\|\,T\exp[-2\|g\|T]}{1-\exp[2\|g\|T]}$$

Since the function $x\exp(-x)/[1-\exp(-x)]$ is decreasing on $[0,\infty]$, one has

$$(43)\quad = \frac{2\delta\|g\|(t-s)}{(1-\exp[-2\|g\|(t-s)])\left[\frac{2\|g\|(t-s)\exp[-2\|g\|(t-s)]}{1-\exp[-2\|g\|(t-s)]}-4\|a\|(t-s)\right]}$$

If we choose $\delta > 0$ so that

$$(44)\quad 4\|a\|\delta T < \frac{2\|g\|\exp[-2\|g\|T]}{1-\exp[-2\|g\|T]},$$

then

$$(45)\quad |P_\delta^t(s)| \le \frac{2\delta\|g\|T}{2\|g\|T\exp[-2\|g\|T]-4\|a\|\delta T(1-\exp[-2\|g\|T])}$$

And from this the first estimate in (36) follows.

To prove the second estimate in (36), one may proceed by differentiating the expression (38). Namely,

$$\frac{\partial v^\varepsilon}{\partial x_i} = E\,\{\frac{\partial\psi}{\partial x_k}\cdot\frac{\partial x_k^\varepsilon(T)}{dx_i}\exp\,[\,\int_t^T h(\frac{x^\varepsilon}{\varepsilon})x^\varepsilon\beta ds]$$

(46)
$$+\ \psi(x^\varepsilon(T))\ (\exp[\int_t^T h(\frac{x^\varepsilon}{\varepsilon})x^\varepsilon\beta ds])$$

$$\cdot\int_t^T (\frac{1}{\varepsilon}\ \frac{\partial h_{jk}}{\partial x_\ell}x_k^\varepsilon + h_{j\ell})\ \frac{\partial x_\ell^\varepsilon}{\partial x_i}\ (s)\ ds\ \}$$

and from (37),

$$d(\frac{\partial x_k^\varepsilon}{\partial x_i}) = (\frac{1}{\varepsilon}\ \frac{\partial g_{kj}}{\partial x_\ell}\ \frac{\partial x_\ell^\varepsilon}{\partial x_i}x_j^\varepsilon + g_{kj}\ \frac{\partial x_j^\varepsilon}{\partial x_i}\ )\ ds$$

(470
$$+\ \frac{1}{\varepsilon}\ \frac{\partial\sigma_{k\ell}}{\partial x_j}\ \frac{\partial x_j^\varepsilon}{\partial x_i}\ db_\ell$$

$$\frac{\partial x_k^\varepsilon}{\partial x_i}(t) = \delta_{ki}\ ,\quad s \geq t \geq 0$$

It follows from (47) that

$$(48)\quad E\,(|\frac{\partial x_k}{\partial x_i}(s)|^4) \leq C[1 + E\int_t^s |x^\varepsilon(r)|^2\,dr]$$

Hence,
$$\leq C\ (1 + |x|^2).$$

$$(49)\qquad E\ (|\frac{\partial x_k^\varepsilon(s)}{\partial x_i}|^2\ ) \leq C\ (1 + |x|)$$

and from this one can readily deduce the second estimate in (36).

Using the function $v^\varepsilon(x,t)$, it is possible to obtain a convenient expression for $p^\varepsilon(t)(\Psi)$.

Proposition 1. Under assumptions (9) we have for any ε, the equality

$$(50) \quad E[p^\varepsilon(T)(\psi)\rho(T)] = E[v^\varepsilon(\xi,0)]$$

$$= \int_{R^n} v^\varepsilon(x,0)\pi_0(x)dx$$

where

$$(51) \quad \pi_0(x) = \frac{1}{[(2\pi)^n \det P_0]^{\frac{1}{2}}} \exp\,[-\tfrac{1}{2}(x-x_0)^T P_0^{-1}(x-x_0)].$$

Proof. From (27) we have

$$(52) \quad E[p^\varepsilon(T)(\psi)\beta(T)] = E[\psi(x^\varepsilon(T))\mu^\varepsilon(T)\beta(T)]$$

$$= E[v^\varepsilon(x^\varepsilon(T),T)\mu^\varepsilon(T)\beta(T)].$$

But

$$(53) \quad dv^\varepsilon(x^\varepsilon(t),t) = \left(\frac{\partial v^\varepsilon}{\partial t} + a_{ij}\frac{\partial^2 v^\varepsilon}{\partial x_i \partial x_j}\right)dt + \frac{\partial v^\varepsilon}{\partial x_i}\sigma_{ij}\,d\tilde{w}_j$$

and

$$(54) \quad d(\mu^\varepsilon\rho) = \rho\mu\,[h(\frac{x^\varepsilon}{\varepsilon})\,x^\varepsilon \cdot dz + \sigma^{-1} g(\frac{x^\varepsilon}{\varepsilon})\,x^\varepsilon \cdot d\tilde{w}] + \rho\mu\beta\cdot dz + \rho\mu\beta\cdot h(\frac{x^\varepsilon}{\varepsilon})x^\varepsilon dt$$

Using this and (35), we have

$$(55) \quad d[v^\varepsilon(x^\varepsilon(t),t)\mu^\varepsilon(t)\rho(t)] = \mu^\varepsilon(t)\rho(t)[(\sigma^*(\frac{x^\varepsilon}{\varepsilon})Dv^\varepsilon(x^\varepsilon(t),t) + v^\varepsilon(x^\varepsilon(t),t)\,\sigma^{-1} g(\frac{x^\varepsilon(t)}{\varepsilon})x^\varepsilon(t)) \cdot d\tilde{w} + v^\varepsilon(x^\varepsilon(t),t)\,(h(\frac{x^\varepsilon(t)}{\varepsilon})x^\varepsilon(t) + \beta(t))\cdot dz].$$

Because of the estimates (36) one can take the expectation of the stochastic integrals obtained by integrating (55). Integrating and taking the expectation gives

$$E\, v^{\varepsilon}(\xi, 0) = E[v^{\varepsilon}(x^{\varepsilon}(T),T)\, \mu^{\varepsilon}(T)\rho(T)]$$

which is the desired result.

QED.

Remark. Note that (50) is well defined if Ψ is Borel bounded and $\beta \in L^{\infty}(0,T;R^d)$. In this case the function v^{ε} is not $C^{2,1}(R^n \times [0,T])$; but this is not essential for the right hand side of (50) to be well defined. Thus, by regularization, it follows that (50) also holds when Ψ is Borel bounded and $\beta \in L^{\infty}(0,T;R^d)$.

7.5. Homogenization

Our objective is to derive a homogenization representation of the conditional distribution $p^{\varepsilon}(t)(\Psi)$ as $\varepsilon \to 0$. We shall begin by considering the homogenization of (35), which is a relatively classical problem. Formally, the method is as follows: We consider an expansion of the form

$$(56)\qquad v^{\varepsilon}(x,t) = v_0(x,t) + \varepsilon v_1(x, \frac{x}{\varepsilon}, t) + \varepsilon^2 v_2(x, \frac{x}{\varepsilon}, t) + \tilde{v}^{\varepsilon}(x,t)$$

Introducing $y = x/\varepsilon$ and using the expression

$$(57)\qquad \frac{\partial}{\partial x_i} \longrightarrow \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}$$

we obtain

$$\frac{\partial v_0}{\partial t} + \frac{\partial v_1}{\partial t} + \varepsilon^2 \frac{\partial v_2}{\partial t} + \frac{\partial \tilde{v}^\varepsilon}{\partial t} + a_{ij}(y) \frac{\partial^2 v_0}{\partial x_i \partial x_j} + g_{ij}(y) x_j \frac{\partial v_0}{\partial x_i}$$

$$+ v_0 h_{ij}(y) x_j \beta_i + \frac{1}{\varepsilon} a_{ij}(y) \frac{\partial^2 v_1}{\partial y_i \partial y_j} + 2\, a_{ij}(y) \frac{\partial^2 v_1}{\partial y_i \partial x_j} + \varepsilon a_{ij}(y) \frac{\partial^2 v_1}{\partial x_i \partial x_j}$$

$$(58) \quad + g_{ij}(y) x_j (\varepsilon \frac{\partial v_1}{\partial x_i} + \frac{\partial v_1}{\partial y_i}) + \varepsilon\, v_i h_{ij}(y) x_j \beta_i$$

$$+ a_{ij}(y) \frac{\partial^2 v_2}{\partial y_i \partial y_j} + 2\varepsilon\, a_{ij}(y) \frac{\partial^2 v_2}{\partial y_i \partial y_j} + \varepsilon^2 a_{ij}(y) \frac{\partial^2 v_2}{\partial x_i \partial x_j}$$

$$+ g_{ij}(y) x_i (\varepsilon^2 \frac{\partial v_2}{\partial x_i} + \frac{\partial v_2}{\partial y_i}) + \varepsilon^2 v_2\, h_{ij}(y) x_j \beta_i - A^\varepsilon \tilde{v}^\varepsilon = 0$$

where we have set

$$(59) \quad A^\varepsilon v = -a_{ij}(y) \frac{\partial^2 v}{\partial x_i \partial x_j} - g_{ij}(y) x_j \frac{\partial v}{\partial x_i} - v\, h_{ij}(y) x_j \beta_i$$

with $y = x/$. We choose

$$(60) \qquad v_1(x,y,t) = v_1(x,t)$$

and

$$(61) \quad \frac{\partial v_0}{\partial t} + a_{ij}(y) \frac{\partial^2 v_0}{\partial x_i \partial x_j} + g_{ij}(y) x_j \frac{\partial v_0}{\partial x_i} + v_0 h_{ij}(y) x_j \beta_i + a_{ij}(y) \frac{\partial^2 v_2}{\partial y_i \partial y_j} = 0$$

To deal with the latter, we introduce $m(y)$ the unique solution of

$$(62) \quad \frac{\partial^2}{\partial y_i \partial y_j} (a_{ij}(y)\, m(y)) = 0$$

m periodic on Y, $m > 0$, $m \varepsilon C^2$, $\int_Y m(y)dy = 1$ (cf. Bensoussan, Lions, and Papanicolaou 1978, p. 530). Then the solvability condition (Fredholm Alternative) for (61) is

$$(63) \quad \begin{aligned} &\frac{\partial v_0}{\partial t} + \bar{a}_{ij} \frac{\partial^2 v_0}{\partial x_i \partial x_j} + \bar{g}_{ij}\, x_j \frac{\partial v_0}{\partial x_i} + v_0 \bar{h}_{ij} x_j \beta_i = 0 \\ &v_0(x,T) = \psi(x), \quad T \geq t \geq 0 \end{aligned}$$

where we have set

$$(64) \quad \bar{a}_{ij} = \int_Y a_{ij}(y)\, m(y) dy$$

and similarly defined $\bar{g}_{ij}$ and $\bar{h}_{ij}$.

If we, in fact, choose

$$(65) \quad v_1(x,t) \equiv 0$$

then $\tilde{v}^{\varepsilon}(x,t)$ is the solution of

$$(66) \quad \begin{aligned} -\frac{\partial \tilde{v}^{\varepsilon}}{\partial t} + A^{\varepsilon}\, \tilde{v}^{\varepsilon} &= \varepsilon \Big(2a_{ij} \frac{\partial^2 v_2}{\partial y_i \partial x_j} + g_{ij} x_j \frac{\partial v_2}{\partial y_i}\Big) \\ &+ \varepsilon^2 \Big(\frac{\partial v_2}{\partial t} + a_{ij} \frac{\partial^2 v_2}{\partial x_i \partial x_j} + g_{ij} x_j \frac{\partial v_2}{\partial x_i} + v_2\, h_{ij} x_j \beta_i \Big) \\ \tilde{v}^{\varepsilon}(x,T) &= 0, \quad T \geq t \geq 0 \end{aligned}$$

To estimate $\tilde{v}^{\varepsilon}$, we proceed as follows: First, we derive an explicit formula for $v_0(x,t)$ which is similar to (38). Consider the Gaussian process

$$(67) \quad d\xi = \bar{g}\xi\, dt + \bar{\sigma} db \quad \xi(t) = x$$

where $\bar{\sigma} = (2\bar{a})^{\frac{1}{2}}$. Using this

$$(68) \quad v_0(x,t) = E\{\psi\,(\xi_{x,t}(T)) \exp[\int_t^T \bar{h}\, \xi_{x,t}(s)\cdot\beta(s)\}$$

and we can easily check that

$$(69) \quad \begin{aligned} |v_0(x,t)| &\leq K_\delta \exp(\delta|x|^2) \\ |Dv_0(x,t)| &\leq K_\delta \exp(\delta|x|^2) \end{aligned}$$

for some K_δ and any $\delta > 0$. An additional calculation shows that

$$(70) \quad \left| \frac{\partial^2 v_0}{\partial x_i \partial x_j} \right| \leq K_\delta \exp(\delta|x|^2).$$

These estimates mean that

$$(70) \quad \left| \frac{\partial v_0}{\partial t} \right| \leq K_\delta \exp(\delta|x|^2).$$

From (61)-(63) we can assert that

---

(1) $\bar{\sigma}$ is not the average of σ. This is a slight abuse of notation.

$$v_2(x,y,t) = \chi_{ij}(y)\frac{\partial^2 v_0}{\partial x_i \partial x_j} + \eta_{ij}(y)x_j\frac{\partial v_0}{\partial x_i} \tag{72}$$
$$+ \; v_0\zeta_{ij}(y)\, x_j\beta_i,$$

for some smooth, bounded functions χ_{ij}, η_{ij}, and ξ_{ij} on Y. Since the higher order derivatives of v_0 also satisfy the bounds (69) - (71), we can deduce from (66) and (72) that

$$-\frac{\partial \tilde{v}^\varepsilon}{\partial t} + A^\varepsilon \tilde{v}^\varepsilon = \varepsilon f^\varepsilon \quad \tilde{v}^\varepsilon(x,T) = 0 \tag{73}$$

where

$$|f^\varepsilon(x,t)| \le K_\delta \exp(\delta|x|^2) \tag{74}$$

Again considering (37), we can write

$$\tilde{v}^\varepsilon(x,t) = \varepsilon E\{\textstyle\int_t^T f^\varepsilon(x^\varepsilon(s),s)(\exp[\int_t^s h(\frac{x^\varepsilon}{\varepsilon})x^\varepsilon\cdot\beta dr])ds\} \tag{75}$$

And, by using arguments similar to those which led to the first estimate in (36), we obtain

$$|\tilde{v}^\varepsilon(x,t)| \le \varepsilon K_\delta \exp(\delta|x|^2) \tag{76}$$

where $\delta > 0$ can be chosen arbitrarily small. Combining this estimate with the expression (72) for v_2, we have proved the following:

Proposition 2. Under the assumption (1) we have the estimate

$$|v^\varepsilon(x,t) - v_0(x,t)| \le \varepsilon K_\delta \exp(\delta|x|^2) \tag{77}$$

where $\delta > 0$ can be chosen arbitrarily small.

By adapting this procedure we can provide a similar analysis of the homogenization properties of the conditional distribution (27) in the nonlinear filtering problem. This is the main result of this section.

First, consider the "limiting filtering problem" defined as follows: Let

$$\begin{aligned} dx &= \bar{g}\, x dt + \bar{\sigma} dw \\ dz &= \bar{h}\, x dt + dv \qquad (78) \\ x(0) &= \xi, \quad z(0) = 0, \quad 0 \le t \le T \end{aligned}$$

and let

$$(79) \quad p^0(T)(\psi) = E[\psi(x(T))\nu^0(T) \mid Z^T]$$

where

$$(80) \quad \nu^0(t) = \exp\{ \int_0^t \bar{h}x \cdot dz - \tfrac{1}{2} \int_0^t |\bar{h}x|^2 ds \}.$$

in which z is a standard Wiener process. ((78) follows from a Girsanov transformation as used in (24)). In fact, we have the well-known formula

$$(81) \quad p^0(T)(\psi) = \exp(-\rho(T)) \cdot \int_{R^n} \frac{\psi(y)\exp[-\tfrac{1}{2}(y-\hat{x}(T))^T P^{-1}(T)(y-\hat{x}(T))]}{(2\pi)^{n/2}(\det P(T))^{\frac{1}{2}}} dy$$

in which

(82) $\rho(t) = \frac{1}{2} \int_0^t |\bar{h}x|^2 \, ds - \int_0^t \bar{h}x \cdot dz$

and $\hat{x}(t)$ is the state of the Kalman filter

$$d\hat{x} = \bar{g}\hat{x}\, dt + P\bar{h}^T (dz - \bar{h}\hat{x}\, dt)$$

$$\hat{x}(0) = x_0$$

(83)

$$\dot{P} + P\bar{h}^T\bar{h}P - \bar{\sigma}\,\bar{\sigma}^T - (\bar{g}P + P\bar{g}^T) = 0$$

$$\bar{P}(0) = P_0$$

As in Proposition 1, we can show that

(84)
$$E[p^0(T)(\psi)\rho(T)] = E[v_0(\xi, 0)] = \int_{R^n} v_0(x,0)\pi_0(x)dx.$$

Using this, we can state the following:

Theorem. Under the assumptions (9) and (33) we have

(85) $$p^\varepsilon(T)(\psi) \xrightarrow[\varepsilon \to 0]{} p^0(T)(\psi)$$

weakly in $L^2(\Omega, Z^T, P)$ for every bounded, uniformly continuous .

Proof. First note that we can assume, without loss of generality, that ψ is smooth and bounded. Indeed, let $\xi_T \varepsilon$ $L^2(\Omega, Z^T, P)$, then using (28)

$$|E[p^{\varepsilon}(T)(\psi)\xi_T| = |E\,\psi(x^{\varepsilon}(T))\mu^{\varepsilon}(T)\xi_T|$$

$$(86) \qquad \leq \|\psi\|_{L^\infty}|\xi_T|_{L^2}\sqrt{E\mu^{\varepsilon}(T)^2}$$

$$\leq C\|\psi\|_{L^\infty}|\xi_T|_{L^2}.$$

Since ψ is uniformly continuous and bounded, it can be approximated in the sup norm by a sequence of smooth, bounded functions. This and the uniform estimate (86) means that it suffices to estabish (85) for smooth ψ's.

Note also that the estimate (86) proves that $p^{\varepsilon}(T)(\psi)$ is bounded in $L^2(\Omega, Z^T, P)$. Therefore, it is sufficient to prove that

$$(87) \qquad E[p^{\varepsilon}(T)(\psi)\rho(T)] \xrightarrow[\varepsilon \to 0]{} E[p^0(T)(\psi)\ \rho(T)]$$

for any β, since the corresponding set of $\rho(T)$'s is dense in $L^2(\Omega, Z^T, P)$, as we have already noted.

But from formulas (50) and (84), the assertion (87) is equivalent to

$$\int_{R^n} v^{\varepsilon}(x,0)\ \pi_0(x)dx \xrightarrow[\varepsilon \to 0]{} \int_{R^n} v_0(x,0)\pi_0(x)dx$$

Since this is immediate from Proposition 2, the Theorem is proved.

Remark. The Theorem implies the convergence of the conditional probability itself in a very weak sense. Indeed we have for any $\xi_T \in L^2(\Omega, Z^T, P)$

$$E^{\varepsilon}\ \pi^{\varepsilon}(T)(\psi)\xi_T = E\psi(x^{\varepsilon}(T))\mu^{\varepsilon}(T)\xi_T = E\ p^{\varepsilon}(T)(\psi)\xi_T$$

$$E^{o}\ \pi^{o}(T)(\psi)\xi_T = Ep^{o}(T)(\psi)\xi_T$$

where $\pi^{o}(T)(\psi) = \dfrac{p^{o}(T)(\psi)}{p^{o}(T)(1)}$

denotes the limit conditional probability, and E^{0} refers to the probability on Ω for which z satisfies (78). Therefore, we can assert that

$$E^{\varepsilon}\pi^{\varepsilon}(T)(\psi)\xi_T \rightarrow E^{0}\pi^{0}(T)(\psi)\xi_T \quad \forall\psi, \forall\xi_T$$

It would be nice to prove stronger convergence results, but it must be kept in mind that the processes (1) themselves converge just in law and not in a stronger sense (cf. Bensoussan, Lions, and Papanicolaou 1978, p. 405).

8. Open Problems and Further Work

The two main issues which we plan to develop in subsequent phases of this research program are the integration of the modeling and control methodologies along the lines initiated in sections 5, 6 and 7, and the development of numerical software to facilitate the application of the integrated methodology to complex structures. The specific issues which we intend to address in Phase II of this research program are:

1. Homogenization and asymptotic analysis of control (including state estimation) for large lattice type structures.

2. Wiener-Hopf - spectral factorization methods for the control of complex space systems with hybrid (lumped and distributed) structure.

3. Development of stabilizing control strategies for nonlinear distributed models, including nonlinear beam and plate models.

4. Synthesis of a design methodology for hybrid nonlinear structures, including the nonlinear differential geometric methods which have been used for finite dimensional control problems and the (linear) methods which we have developed for the treatment of distributed linear models.

1. Asymptotic Analysis and Homogenization of Control Problems for Lattice Type Structures.

In Phase I of this program the use of homogenization in connection with control system designs was demonstrated in the analysis of two abstract control and filtering problems. This analysis established the mathematical feasibility of the technique. Previously, homogenization had been used only for model reduction, and it had not been applied in a control theoretic setting. Since the basic optimization techniques, like the Bellman equation, are inherently nonlinear, it was not clear how the methodology could be used. The feasibility of the homogenization methodology has now been demonstrated in the context of abstract control problems.

We have also demonstrated use of the methodology in the construction of simplified models for certain kinds dynamical phenomena propagating on lattice structures. We have treated heat conduction type problems, by exploiting the connection between such problems and an associated probabilistic structure, and simple one dimensional lattice structures using purely analytical (PDE) methods for model simplification. The asymptotic analysis method involves a study of the convergence of the resolvents of certain operators using the theory of "correctors" introduced for this class of problems by Bensoussan, Lions, and Papanicolaou (1978). The method handles the transition from the "discrete" operators characterizing lattice type structures to the PDE operators characterizing the continuum approximations of the structures. It also produces the natural continuum model in the course of the asymptotic analysis; that is, it is not necessary to postulate the model a priori. For example, homogenization of the one dimensional lattice structure in section 5 produced the Timoshenko beam model rather than the Euler beam model which one might have expected from the symmetry of the orginal formulation.

In the second phase of this program we shall develop the methodology to treat the combined problem of modeling and control of lattice type structures; that is, we shall develop the homogenization - optimization methodology described in section 6 to treat realistic models of the dynamical control of large lattice and plate structures.

2. Wiener-Hopf - spectral factorization methods for the control of complex space systems with hybrid (lumped and distributed) structure.

Thus far we have applied our control theory only to purely distributed models of a very simple character. While it is clear that the methods can be used for the design and analyis of control systems for structures with both distributed and lumped components, it would be useful to treat a problem including both kinds of elements with an overall linear model. The NASA challenge problem is of this type and we shall consider it, adapting our frequency domain methods as required to carry out the design. Since this problem is being considered by several researchers using a variety of methods, comparison of results and capabilities will be possible. While there are no conceptual problems in this extension of our methods, we do expect to encounter challenging numerical problems.

3. **Development of stabilizing control strategies for nonlinear distributed models, including nonlinear beam and plate models.**

Many of the applications of large space structures require maneuvers which cannot be faithfully described by linear models. For example, large attitude excursions of telescope and antenna structures require equations for the evolution of the Euler angles through the course of the maneuver. We plan to extend our methods to treat certain aspects of this class of problems. It will be necessary to use differential geometric methods to describe the global dynamics of the system undergoing large angle maneuvers. Recent work by Baillieul (1983), El Baraka and Krishnaprasad (1984), among others has led to a theory for the attitude dynamics for articulated structures. Elements of this theory in combination with the methods for the control of distributed systems which have been used in Phase I of this project should be a useful starting point for the development of a comprehensive theory for large scale motions of complex, distributed structures.

Specific issues to be addressed include the use of stabilization techniques for semilinear distributed systems. These are systems in which the controls enter the dynamics by multipling the state. Common examples include the dynamics of a beam in which the applied load can be manipulated. When the load is used as a feedback control, the system is nonlinear, and the theory of nonlinear semigroups is the most convenient setting for the analysis. In a series of papers Ball and Slemrod have derived conditions for the stabilization of such systems. In particular, they show that stabilization of the Euler beam by semilinear feedback is a delicate problem, and that the most natural conditions tend to lead to a weak form of stability.

In the second phase of this program we plan to combine this theory with the corresponding theory for the stabilization of finite dimensional nonlinear systems undergoing large attitude motions.

4. Synthesis of the Nonlinear and Distributed Design Methods

Resolution of the problems in 1.-3., will lead to a design procedure for (a class of) control systems for large space structures. Work will be necessary to unify the various methods into a software system for computer-aided-design. We shall use software systems for symbolic manipulation (either MACSYMA or SMP) to implement the complex analysis involved in the initial reduction of the modeling and control problem (for example, by carrying out an asymptotic analysis in the context of the control system design). This will permit us to base the selection of numerical routines for the implementation of the models and control laws on simplified structural models. This will in turn reduce the number and diversity of costly computer runs which must be made with conventional design tools.

References

B.D.O. Anderson 1967, An algebraic solution to the spectral factorization problem, IEEE Trans. Automatic Control, AC-12, pp. 410-414.

M. Anderson 1981, Buckling of periodic lattice structures, AIAA Journal, 19, pp. 782-788.

M. Aswani 1982, Development of an Analytical Model for Large Space Structures, DTIC Report ADA 119349.

I. Babuska 1975, Homogenization and its applications. Mathematical and computational problems, Technical Note BN-821, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park.

M.J. Balas 1978a, Modal control of certain flexible dynamic systems, SIAM J. Control, 16, pp. 450-462.

M.J. Balas 1978b, Feedback control of flexible systems, IEEE Trans., AC-23, pp. 673-679.

M.J. Balas 1982, Trends in large space structure control theory: fondest hopes, widest dreams, IEEE Trans., AC-27, pp. 522-535.

J. Baillieul 1978, Geometric methods for nonlinear optimal control problems, J. Opt. Theory. Appl., 25, pp. 519-548.

J. Baillieul 1981, Controllability and observability of polynominal dynamical systems, Nonlinear Analysis: Theory, Methods, and Appl., 5, pp. 543-552.

J. Baillieul 1983, Modelling and control of flexible and articulated spacecraft, preprint.

A. Bensoussan 1979, Homogenisation theory, Conf. del Seminaria de Math. dell'Univ. de Bari, No. 158.

A. Bensoussan and J.L. Lions 1978, Applications des Inequations Variationelles en Controle Stochastique, Dunod, Paris.

A. Bensoussan, J.L. Lions, and G.C. Papanicolaou 1978, Asymptotic Analysis for Periodic Structures, North Holland, Amsterdam.

G.L. Blankenship 1979, Asymptotic analysis in mathematical physics and control theory: Some problems with common features, Richerchi di Automatica, vol. 10, pp. 265-315.

J.F. Bourgat 1978, Numerical Experiments of the Homogenization Method for Operators with Periodic Coefficients, INRIA Report No. 277.

L. Brieman 1968, Probability, Addison-Wesley, Reading, Mass.

R.W. Brockett 1970, Finite Dimensional Linear Systems, John Wiley, New York.

S.H. Crandall, et al. 1979, Dynamics of Mechanical and Electromechanical Systems, McGraw-Hill, New York.

R.F. Curtain 1976, Estimation theory for evolution equations excited by general white noise, SIAM J. Control Optim., 14, pp. 1124-1150.

R.F. Curtain and A.J. Pritchard 1976, The infinite-dimensional Riccati equation for systems defined by evolution equations, SIAM J. Control Optim., 14, pp. 951-983.

R. Datko 1971, A linear control problem in abstract Hilbert space, J. Diff. Equs., 9, pp. 343-359.

R. Datko 1970, Extending a theorem of A.M. Lyapunov to Hilbert space, J. Math. Anal. Appl., 32, pp. 610-616.

J.H. Davis, 1978, A distributed filter derivation without Riccati equations, SIAM J. Control Optim., 16, pp. 584-592.

J.H. Davis 1979a, Models and filters for travelling wave disturbance problems, IEEE Trans. Automatic Control, AC-24, pp. 283-289.

J.H. Davis 1979b, Wiener-Hopf methods for open-loop unstable distributed systems, SIAM J. Control Optim., 17, pp. 713-728.

J.H. Davis 1982, Hybrid models for state estimation in unit train control, IEEE Trans. Automatic Control, AC-27, pp. 306-319.

J.H. Davis and B.M. Barry 1977, A distributed model for stress control in multiple locomotive trains, Appl. Math. Optim., 3, pp. 163-190.

J.H. Davis and R.G. Dickinson 1983, Spectral factorization by optimal gain iteration, SIAM J. Appl. Math., 43, pp. 289-301.

D.L. Dean and S. Tauber 1959, Solutions for one dimensional lattice structures, J. Eng. Mech. Div., ASCE, 85, pp. 31-41.

M.C. Delfour 1974, Numerical solution of the optimal control problem for linear hereditary differential systems with a linear- quadratic cost function and approximation of the Riccati differential equation, Technical Report CRM-408, Centre de Recherches Mathematiques, Universite de Montreal.

M. El-Baraka and P.S. Krishnaprasad 1984, Geometric methods for multibody dynamics, Proc. AIAA Dynamics Specialists' Conf., Palm Springs, May 1984, AIAA Paper No. 84-1022-CP.

M. El-Baraka 1983, Nonlinear Invertibility and Disturbance Decoupling of Altitude Control Systems, M.S. Thesis, Elec. Engrg. Dept., Univ. of Maryland, College Park.

A. Feintuch and M. Rosenfeld 1978, On pole assignment for a class of infinite dimensional systems SIAM J. Control Optim., 16, pp. 270-276.

S. Geman 1982, Almost sure stable oscillations in a large system of randomly coupled equations, SIAM J. Appl. Math., vol. 42, pp. 695-703.

J.S. Gibson 1979, The Riccati integral equations for optimal control problems on Hilbert Spaces, SIAM J. Control Optim., 17, pp. 537-565.

J.S. Gibson 1980, A note on stabilization of infinite dimensional linear oscillators by compact feedback, SIAM J. Control Optim., 18, pp. 311-316.

I.C. Gohberg and M.G. Krein 1960, Systems of integral equations with kernel depending on the difference of the arguments, Amer. Math. Soc. Transl., 14, pp. 217-287.

I.C. Gohberg and M.G. Krein 1962, Integral equations on the half-line with kernel depending on the difference of the arguments, Amer. Math. Soc. Transl., 22, pp. 163-288.

F. Gruber and M.M. Bayoumi 1982, Suboptimal control strategies for multilocomotive power trains, IEEE Trans. Automatic Control, AC-27, pp.

J.M. Greenberg, R.C. MacCamy, and V.J. Mizel 1968, On the existence, uniqueness, and stability of solutions of the equation d'(u)U + U = , J. Math. Mech., 17, pp. 707-728.

E. Hille and R.S. Phillips 1957, Functional Analysis and Semigroups, Colloquium Publications, vol. 31, American Math. Soc., Providence.

J.W. Helton 1976, A spectral factorization approach to the distributed stable regulator problem: the algebraic Riccati equation, SIAM J. Control, 14, pp. 639-661.

T. Kato 1966, Perturbation Theory for Linear Operators, Springer-Verlag, New York.

S. Kirkpatrick 1973, Perculation and conduction, Rev. Mod. Phys., 45, pp. 574-588.

C. Kittel 1976, Introduction to Solid State Physics, Wiley, New York.

J.B. Keller 1977, Effective behavior of heterogeneous media, in Statistical Mechanics and Staistical Methods in Theory and Application, U. Landman, ed., Plenum, New York, pp. 613-644.

R. Kunnemann 1983, The diffusion limit for reversible jump processes on Z with ergodic random bond conductivities, Commun. Math. Phys., 90, pp. 27-68.

J.H. Lang and D.H. Staelin 1982a, Electrostatically controlled large aperature reflecting satellite antennas, IEEE Trans. Automatic Control, AC-27, pp. 666-670.

J.H. Lang and D.H. Staelin 1982b, The computer-controlled stabilization of a noisy two-dimensional hyperbolic structure, IEEE Trans. Automatic Control, AC-27, pp. 1033-1043.

E. Larsen 1975 1976, Neutron transport and diffusion in inhomogeneous media, I, J. Math. Phys., vol. 16, pp. 1421-1427; II, Nucl. Sci. Eng., vol. 60, pp. 357-368.

J.L. Lions 1971, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, New York.

D.L. Lukes and D.L. Russell 1969, The quadratic criterion for distributed systems, SIAM J. Control, 7, pp. 101-121.

P.J. McLane, L. Peppard, and M. Sundareswaran 1973, Decentralized feedback control for the brakeless control of multi-locomotive trains, IEEE Trans. Automatic Control, AC-21, pp. 358-363.

M.M. Mikulus 1978, Structural efficiency of long lightly loaded truss and isogrid columns for space application, NASA TM 78687.

B.P. Molinari 1973a, The stabilizing solution of the algebraic Riccati equation, SIAM J. Control, 11, pp. 262-271.

B.P. Molinari 1973b, Equivalence relations for the algebraic Riccati equation, SIAM J. Control, 11, pp. 272-285.

A. Nayfeh and M.S. Hefzy 1978, Continuum modeling of three-dimensional truss-like space structures, AIAA Journal, 16, pp. 779-787.

A.K. Noor, M.S. Anderson, and W.H. Greene 1978, Continuum models for beam- and plate-like lattice structures, AIAA Journal, 16, pp. 1219-1228.

I.S. Pace and S. Barnett 1972, Comparison of numerical methods for solving Lyapunov matrix equations, Int. J. Control, 15, pp. 907-915.

J.D. Renton 1984, The beam-like behavior of space trusses, AIAA Journal, 22, pp. 273-280.

J.D. Renton 1973, Buckling of long, regular trusses, Int. J. Mech. Sci., 9, pp. 1489-1500.

J.D. Renton 1970, General properties of space grids, Int. J. Mech. Sci., 12, pp. 801-810.

J.D. Renton 1969, Behavior of Howe, Pratt and Warren trusses, J. Struc. Div., ASCE., 95, pp. 185-202.

D.L. Russell 1972, Control theory of hyperbolic equations related to certain questions in harmonic analysis and spectral theory, J. Math. Anal. Appl., 40, pp. 336-368.

F. Stenger 1972, The approximate solution of Wiener-Hopf integral equations, J. Math. Anal. Appl., 37, pp. 687-724.

C. Tai 1971, Dyadic Green's Functions in Electromagnetic Theory, Intext, San Francisco.

L.W. Taylor, Jr. and A.V. Balakrishnan 1984, A mathematical problem and a spacecraft control laboratory experiment (SCOLE) used to evaluate control laws for flexible spacecraft, NASA/IEEE design challenge, Spacecraft Control Branch, NASA Langley Research Center, Hampton, VA.

R. Triggiani 1975, On the stabilizability problem in Banach space, J. Math. Anal. Appl., 52, pp. 383-403.

R.B. Vinter 1977, Filter stability for stochastic evolution equations, SIAM J. Control Optim., 15, pp. 465-485.

R.B. Vinter and T. L. Johnson 1977, Optimal control of nonsymmetric hyperbolic systems in n variables on the half-space, SIAM J. Control Optim., 15, pp. 129-143.

N. Weiner and E. J. Akutowicz 1969, A factorization of positive Hermitian matrices, Math. Mech., 8, pp. 111-120.

J.C. Willems 1971, Least squares stationary optimal control and the algebraic Riccati equation, IEEE Trans. Automatic Control, AC-16, pp. 621-634.

G. Tunnicliffe Wilson 1972, The factorization of matricial spectral densities, SIAM J. Appl. Math., 23, pp. 420-426.

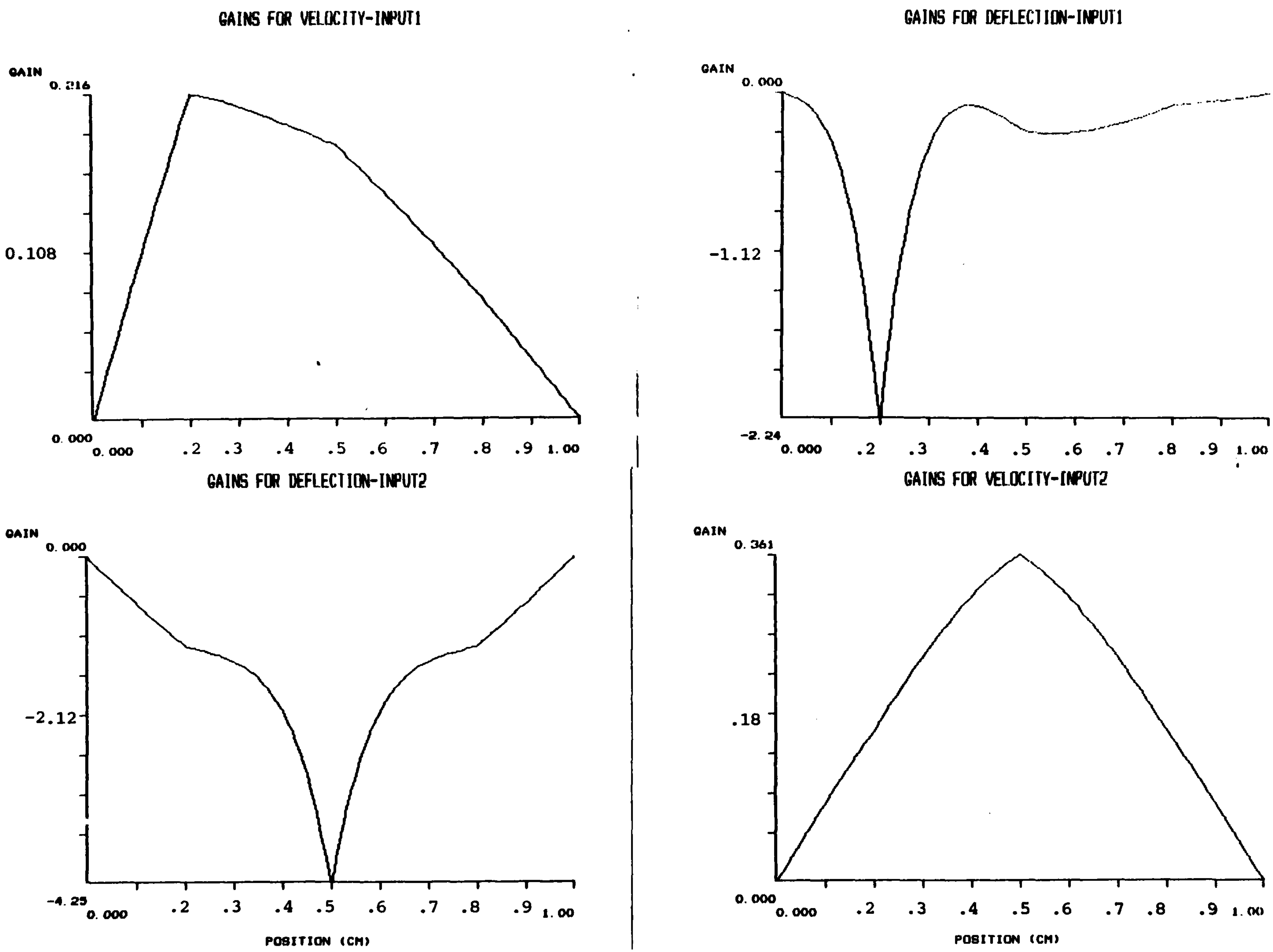
G. Tunnicliffe Wilson 1978, A convergence theorem for spectral factorization, J. Multivar. Anal., 8, pp. 222-232.

D.C. Youla 1961, On the factorization of rational matrices, IRE Trans. Information Theory, IT-7, pp. 172-189.

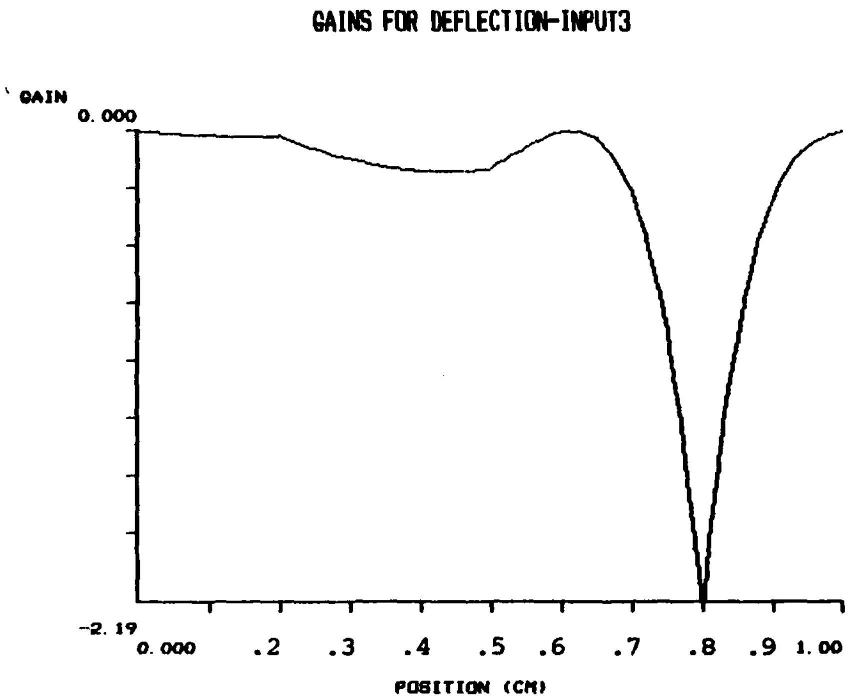
APPENDIX A

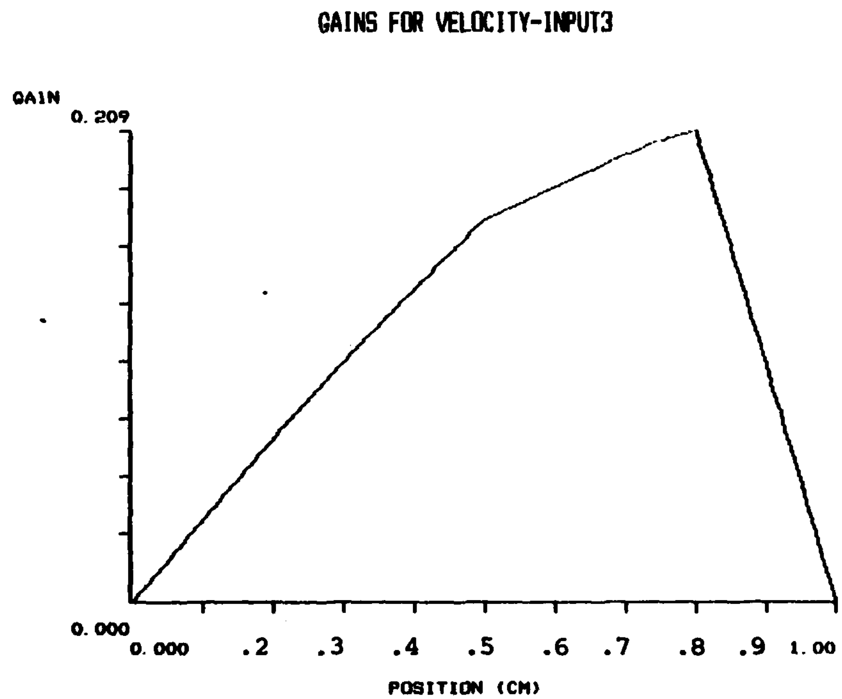
Experimental Results For The Euler Beam

The plots appear in groups. A first page of each group contains parameters of the run. For example the first group is preceeded by a page with the following parameters. CONTROL: .2,.5.,.8. The number designate locations of point actuators. Similar remarks apply for the OBSERV, where this OBSERV designates the points on the beam whose displacement is penalized in the cost criterion. FREQUENCY RANGE is the range of frequency over which we evaluate the transfer function matrix, the spectral factors, and the resolvent. RELATIVE WEIGHT OF X VS U is the weight *of the norm of* observation vector, assuming the weight of the control vector to be one. The name of the file is a working variable. WHAT MODE: here we list all the spacial modes that are used to displace the beam (one at a time), and the next line provides the amplitude of this disturbance. FEEDBACK GAIN: is -1 for the optimal gain and it is 0 for the open loop. The meaning of the plot titles is as follows: "Beam at x, yth mode," means that the beam is initially displaced by the y-th spacial mode, and the deflection of the beam is observed as a function of time at point x. On the control plot we indicate the position of the point control and the time evolution of the control at that point.

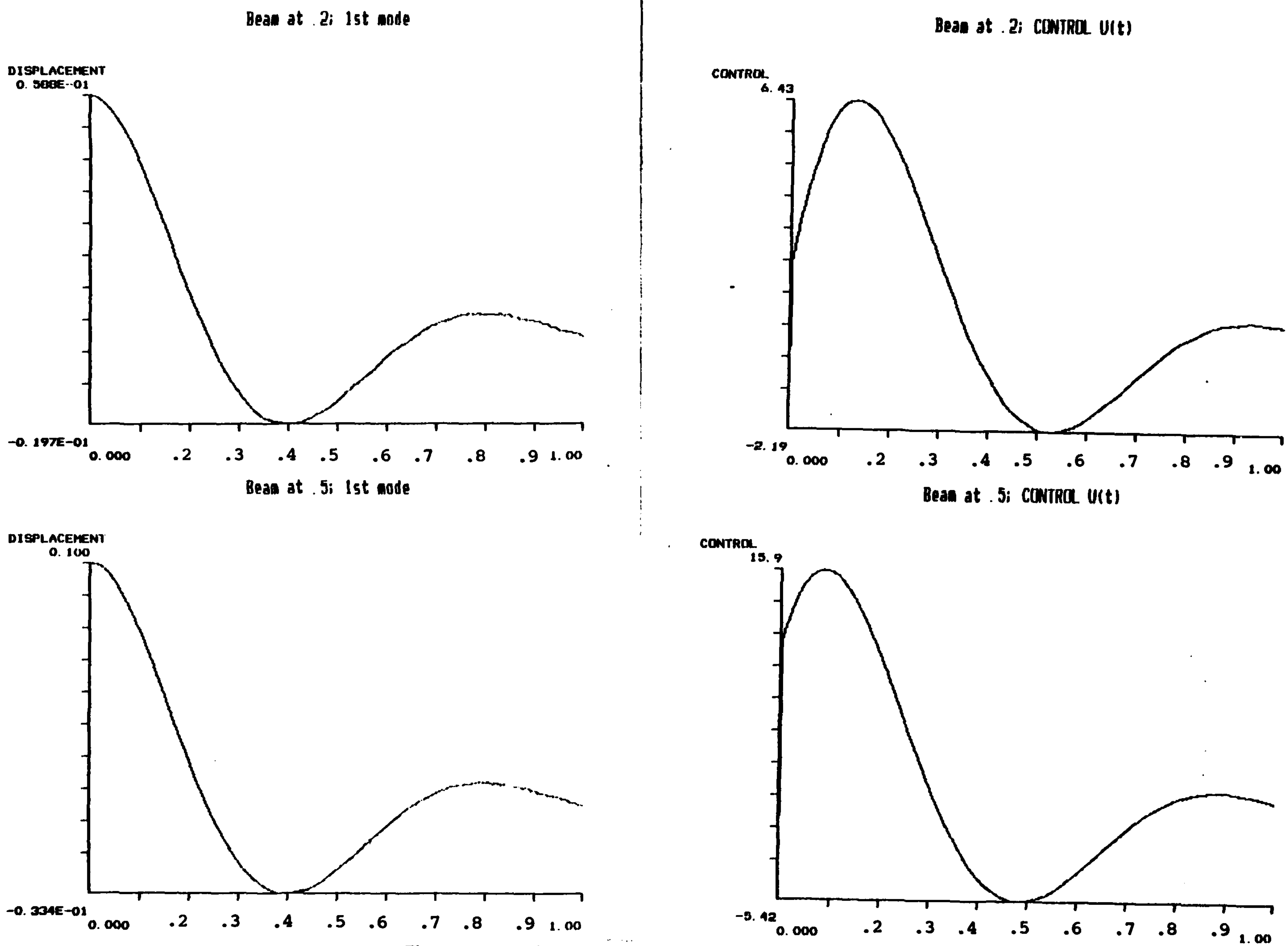


Three inputs/outputs at .2, .5 and .8.

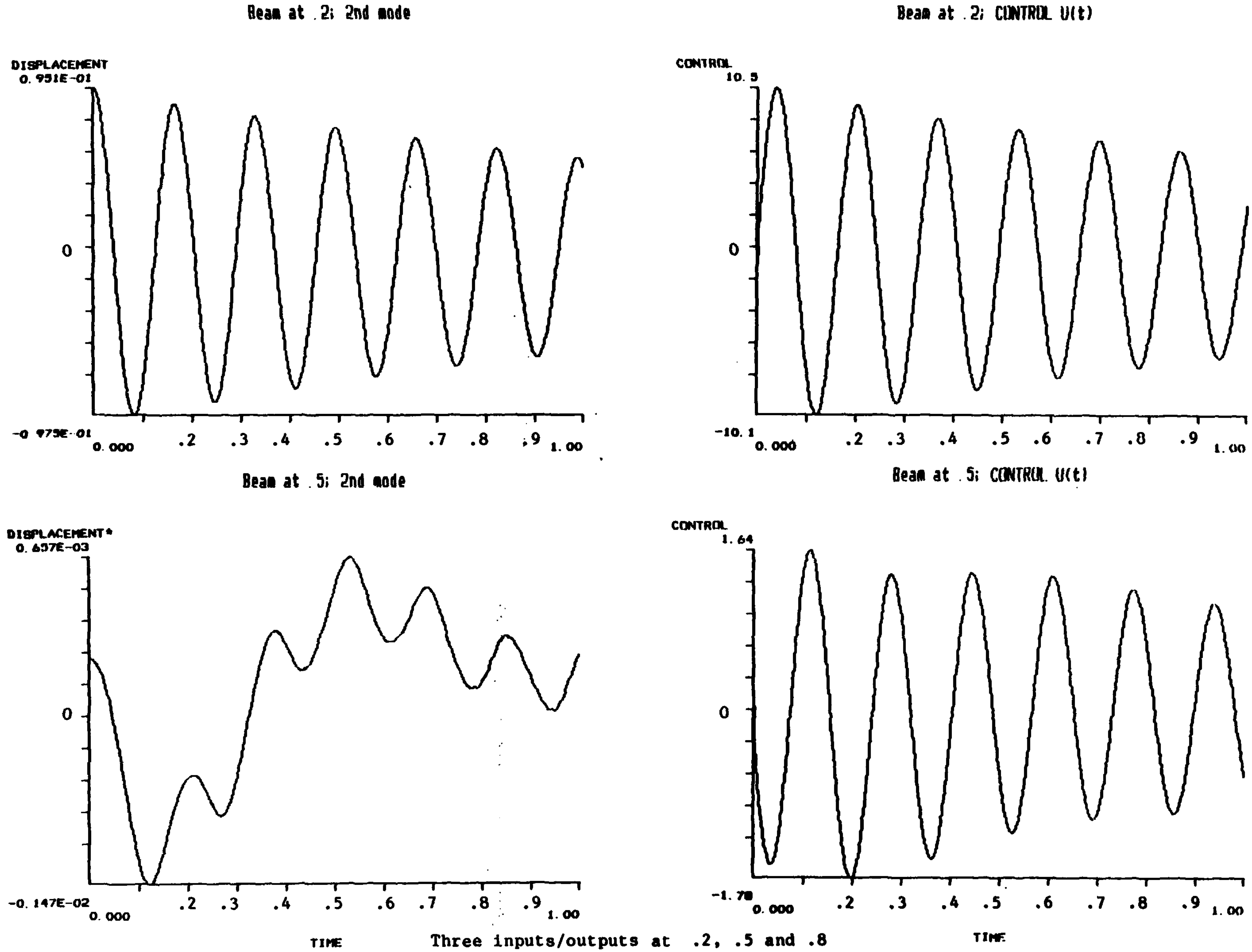




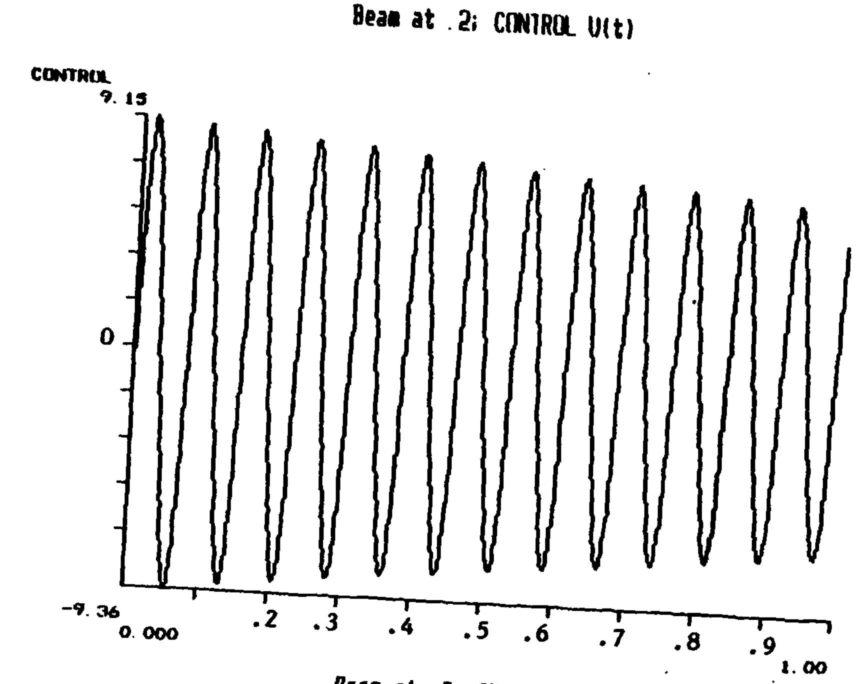
Three inputs/outputs at .2, .5 and .8

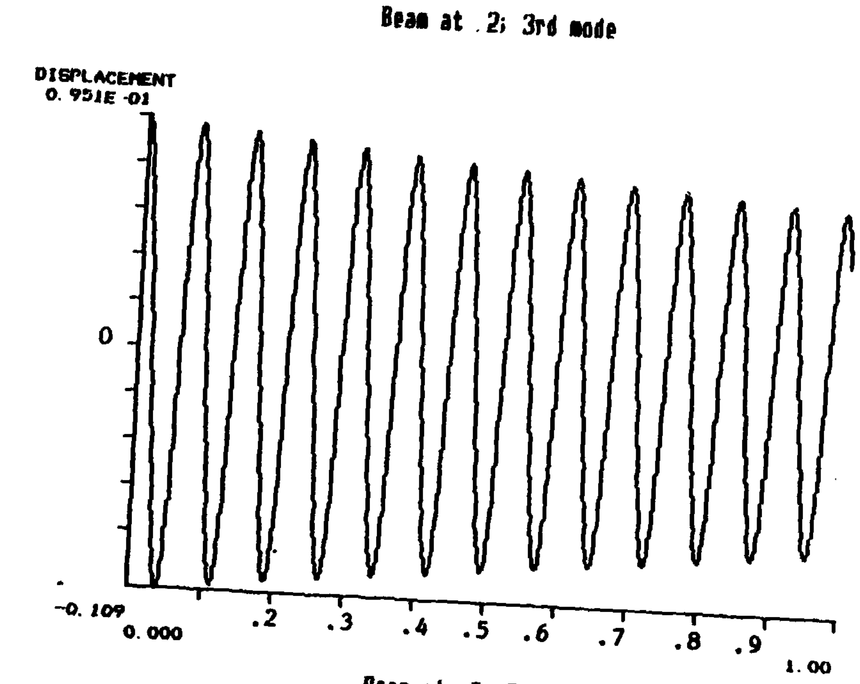


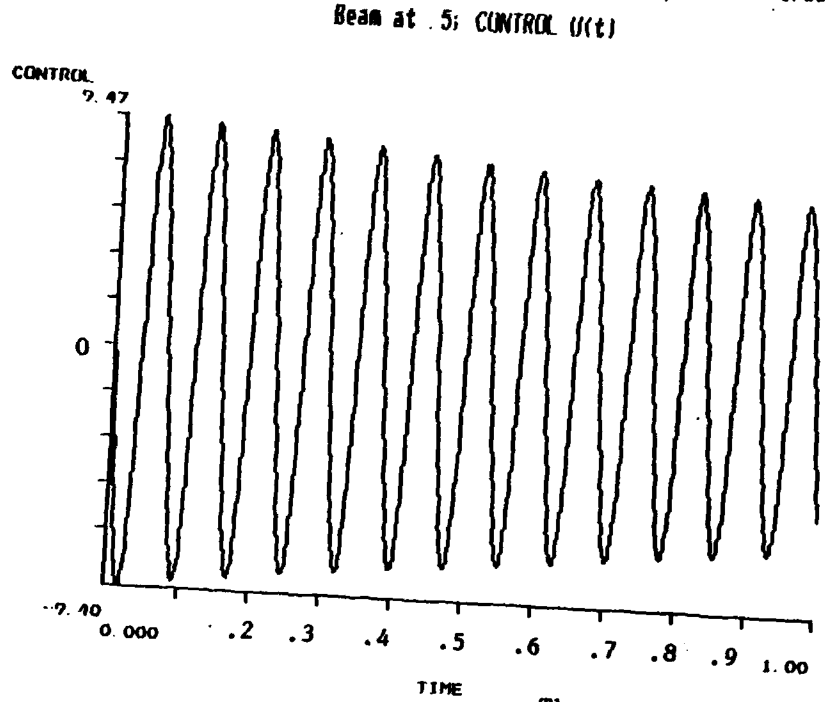
Three inputs/outputs at .2, .5 and .8

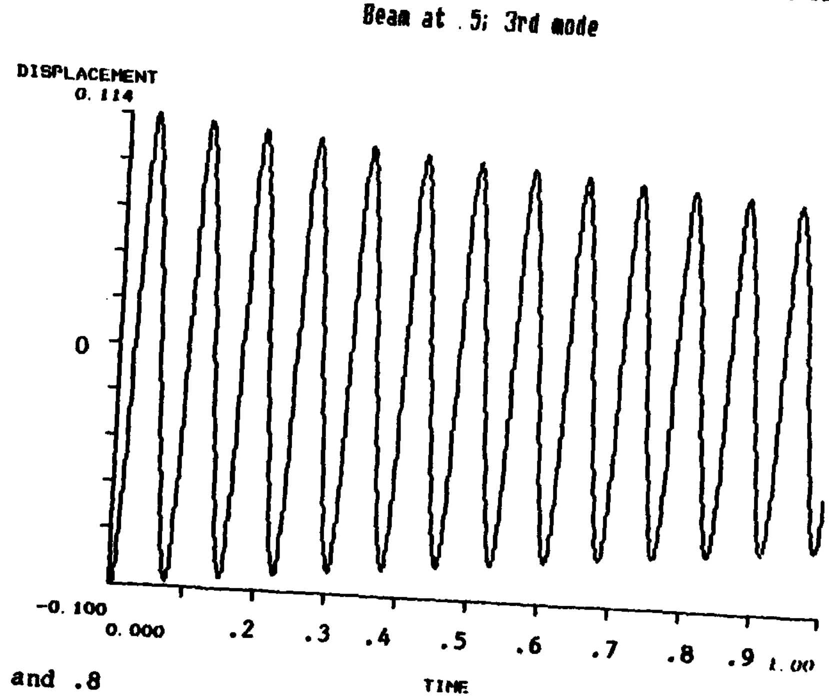


Three inputs/outputs at .2, .5 and .8

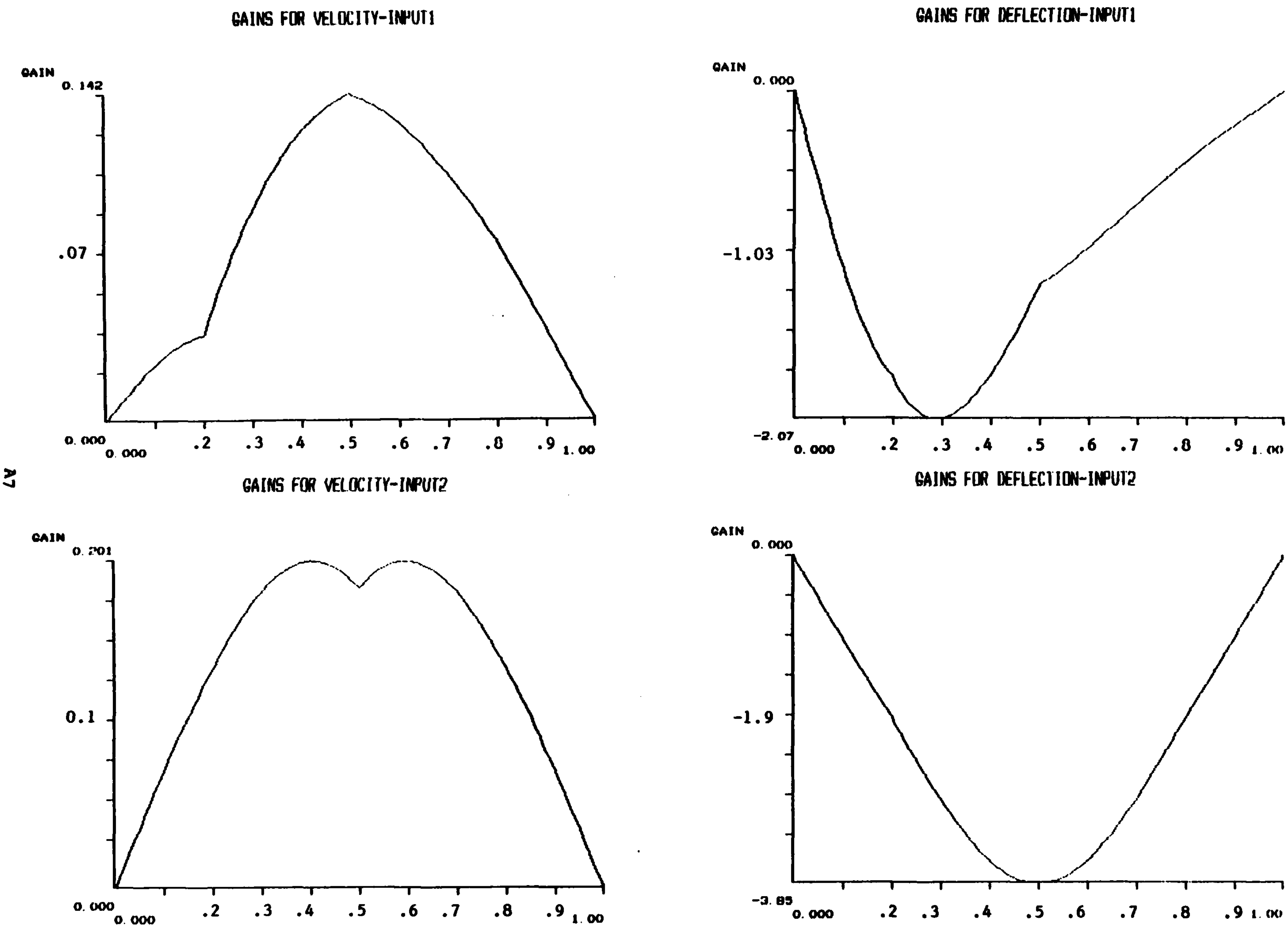




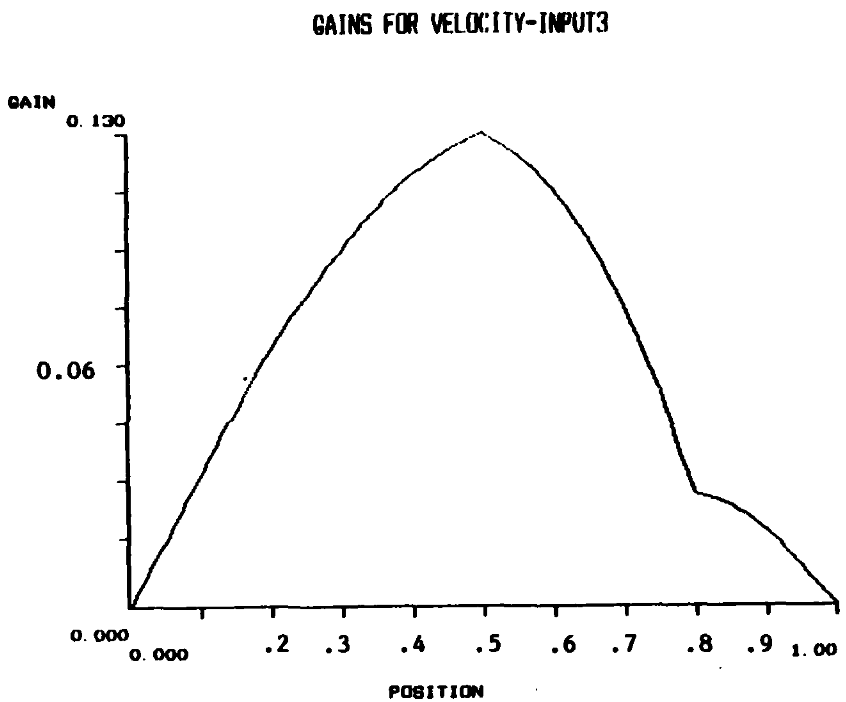


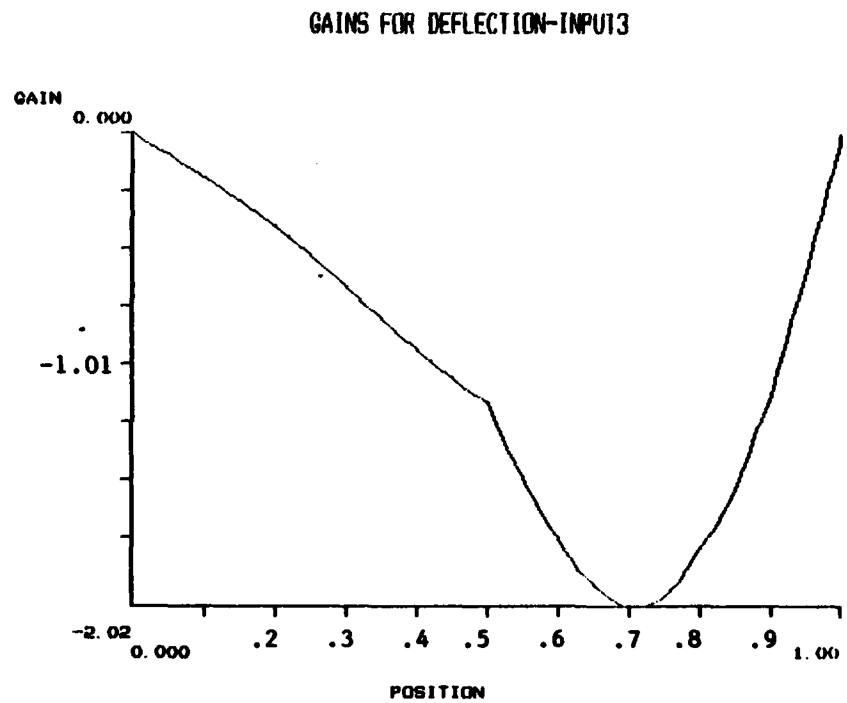


Three inputs/outputs at .2, .5 and .8

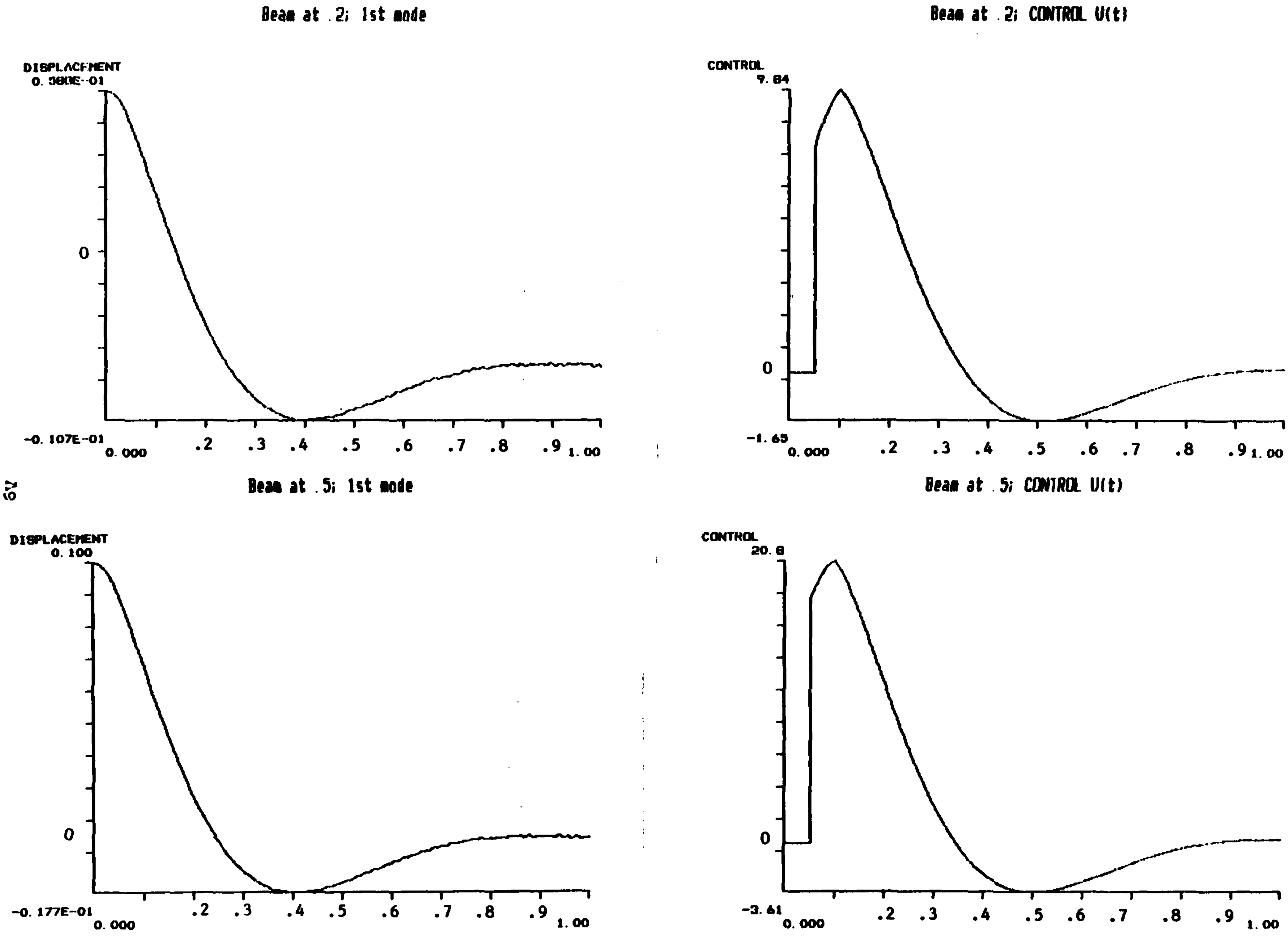


Three inputs/outputs at .2, .5 and .8; delay = .050

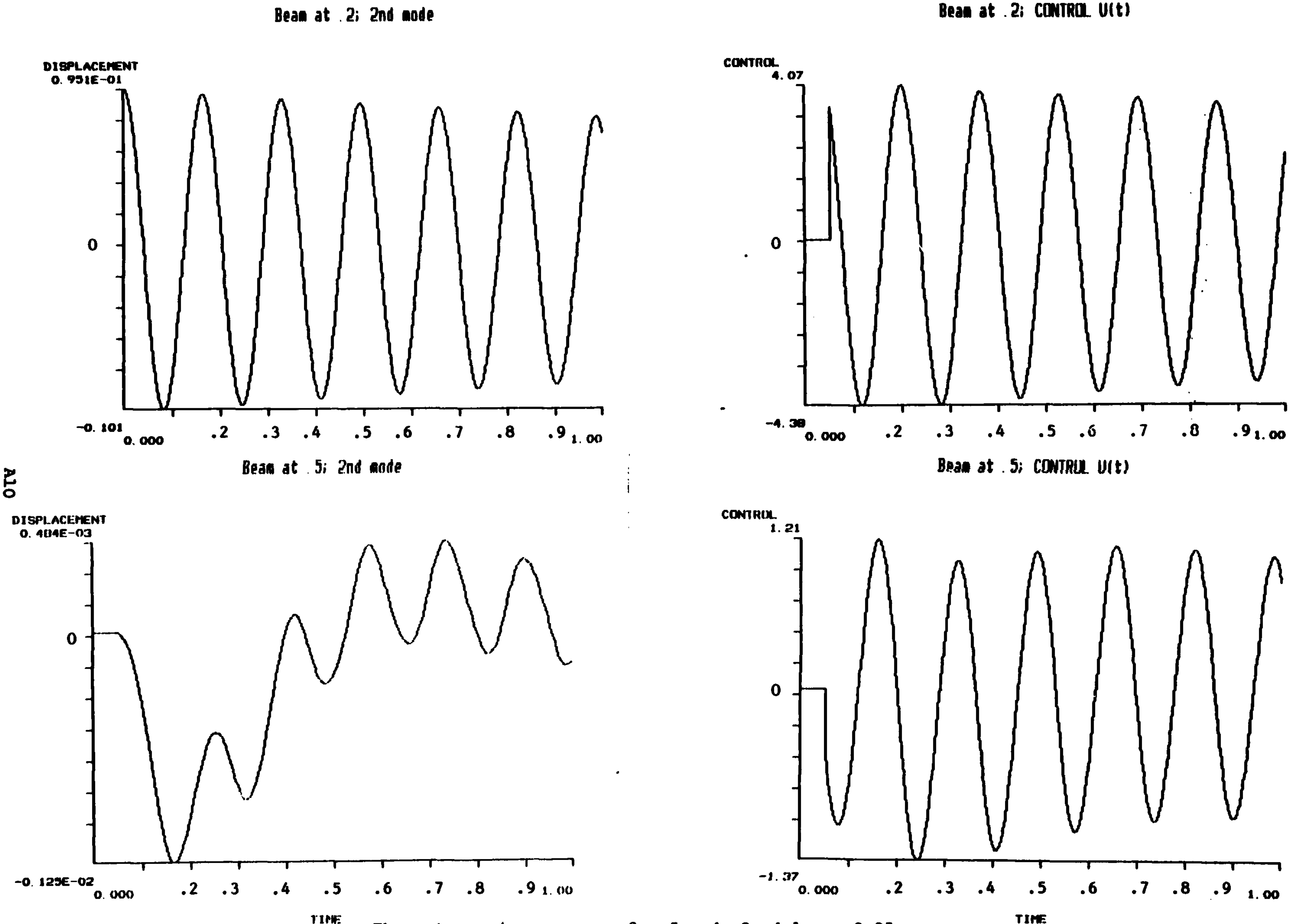




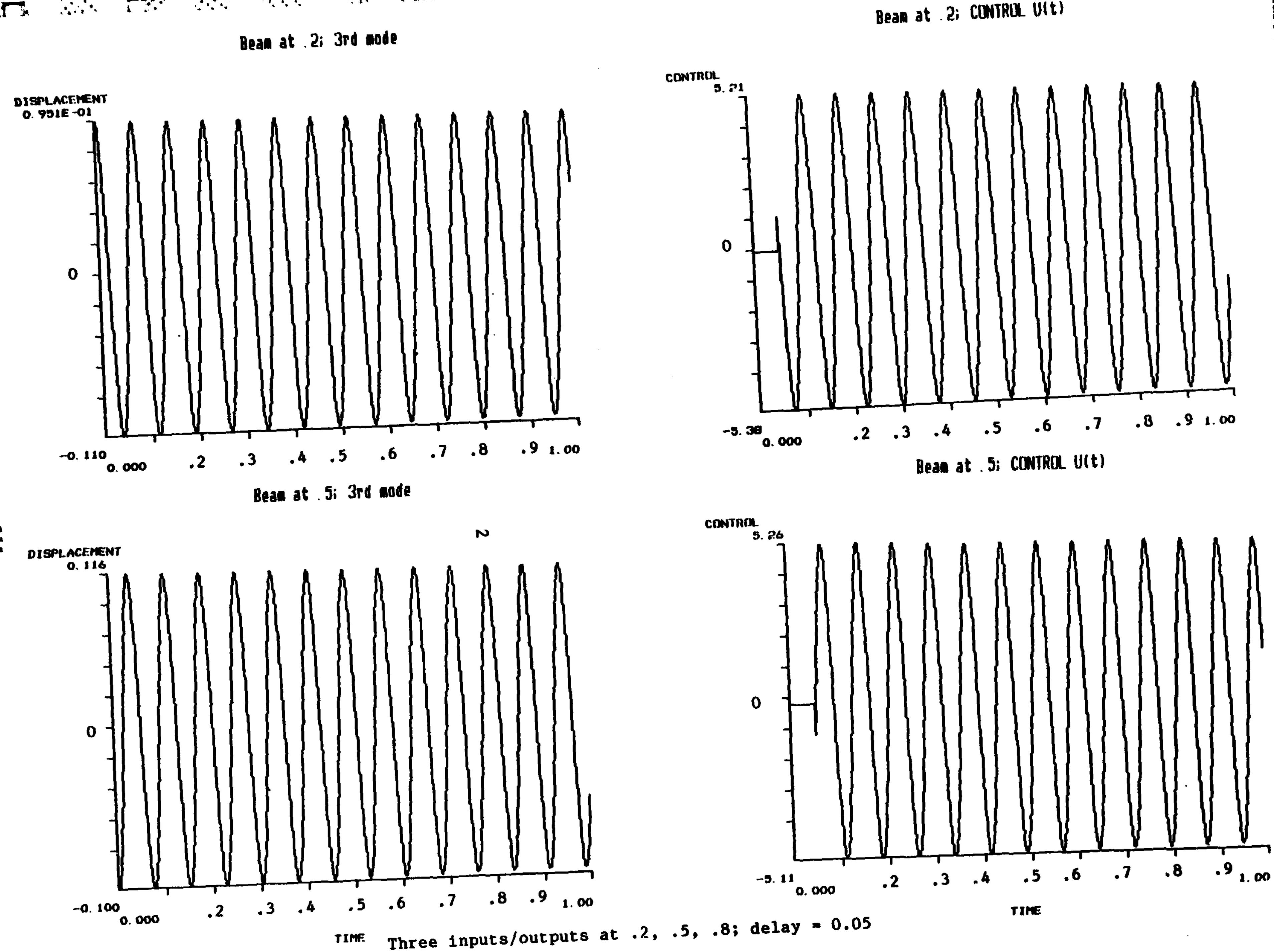
Three inputs/outputs at .2, .5 and .8; delay = .050



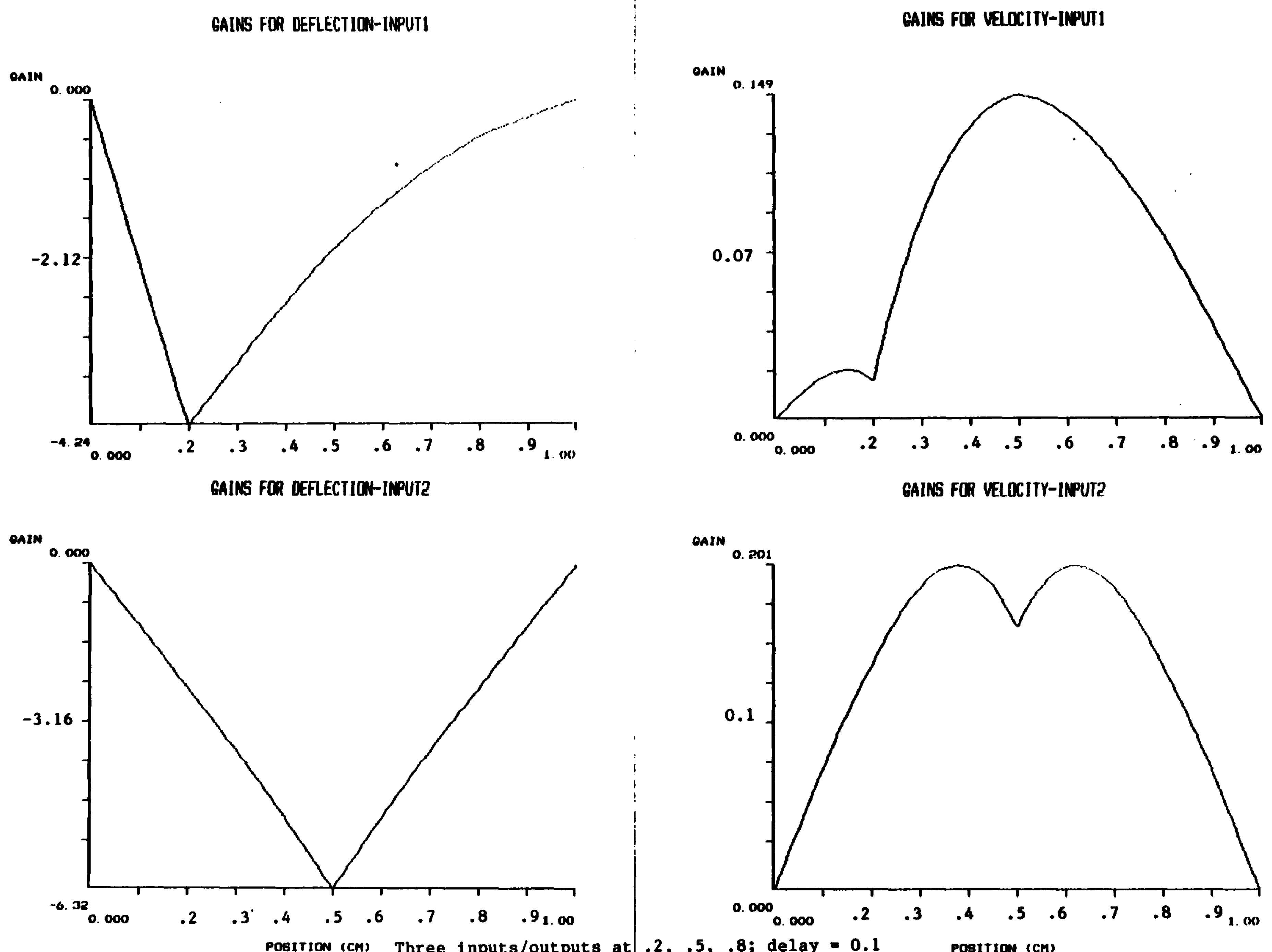
Three inputs/outputs at .2, .5, .8; delay = .050



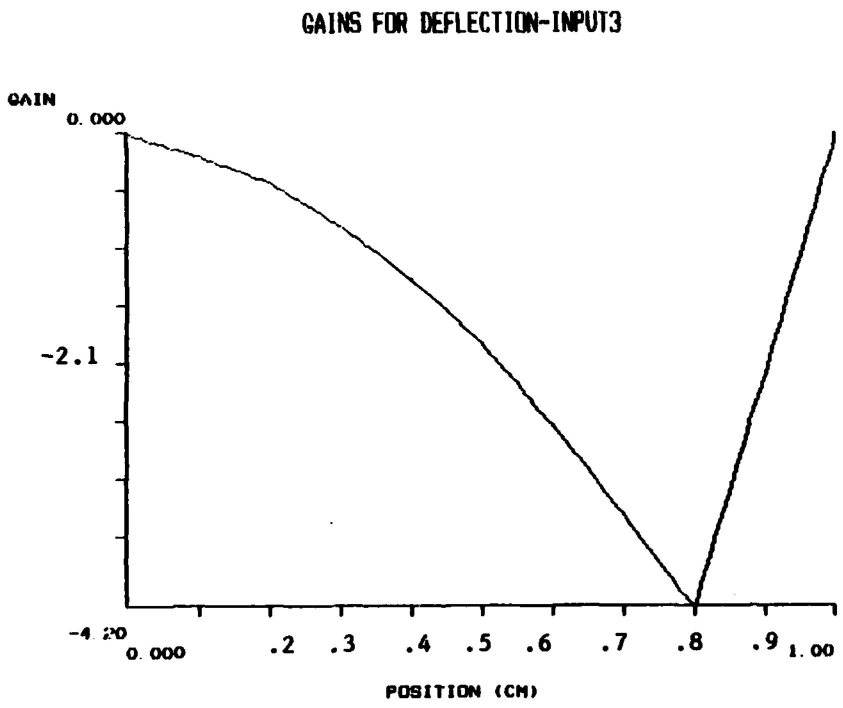
Three inputs/outputs at .2, .5 and .8; delay = 0.05

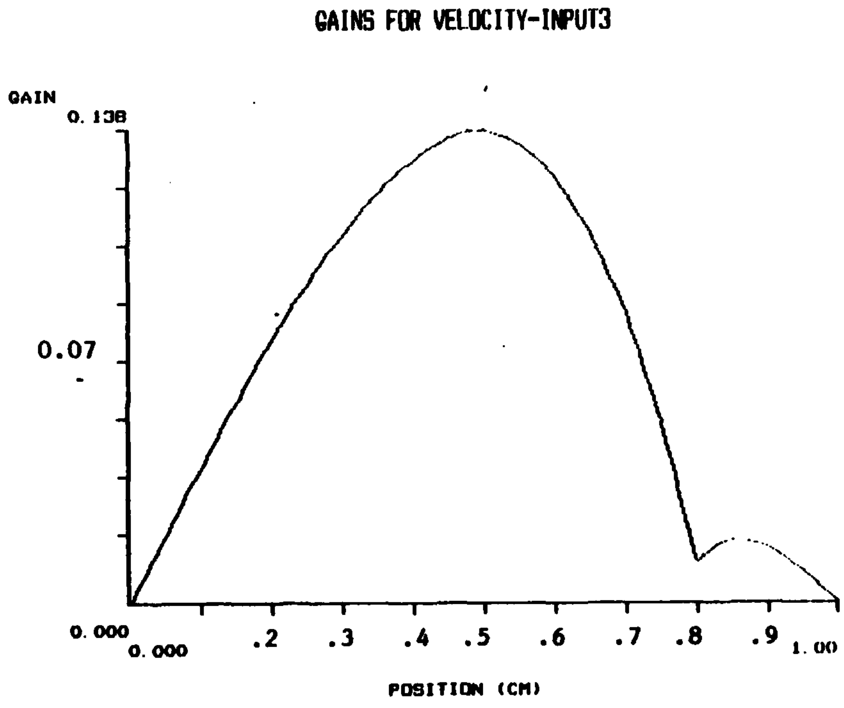


Three inputs/outputs at .2, .5, .8; delay = 0.05

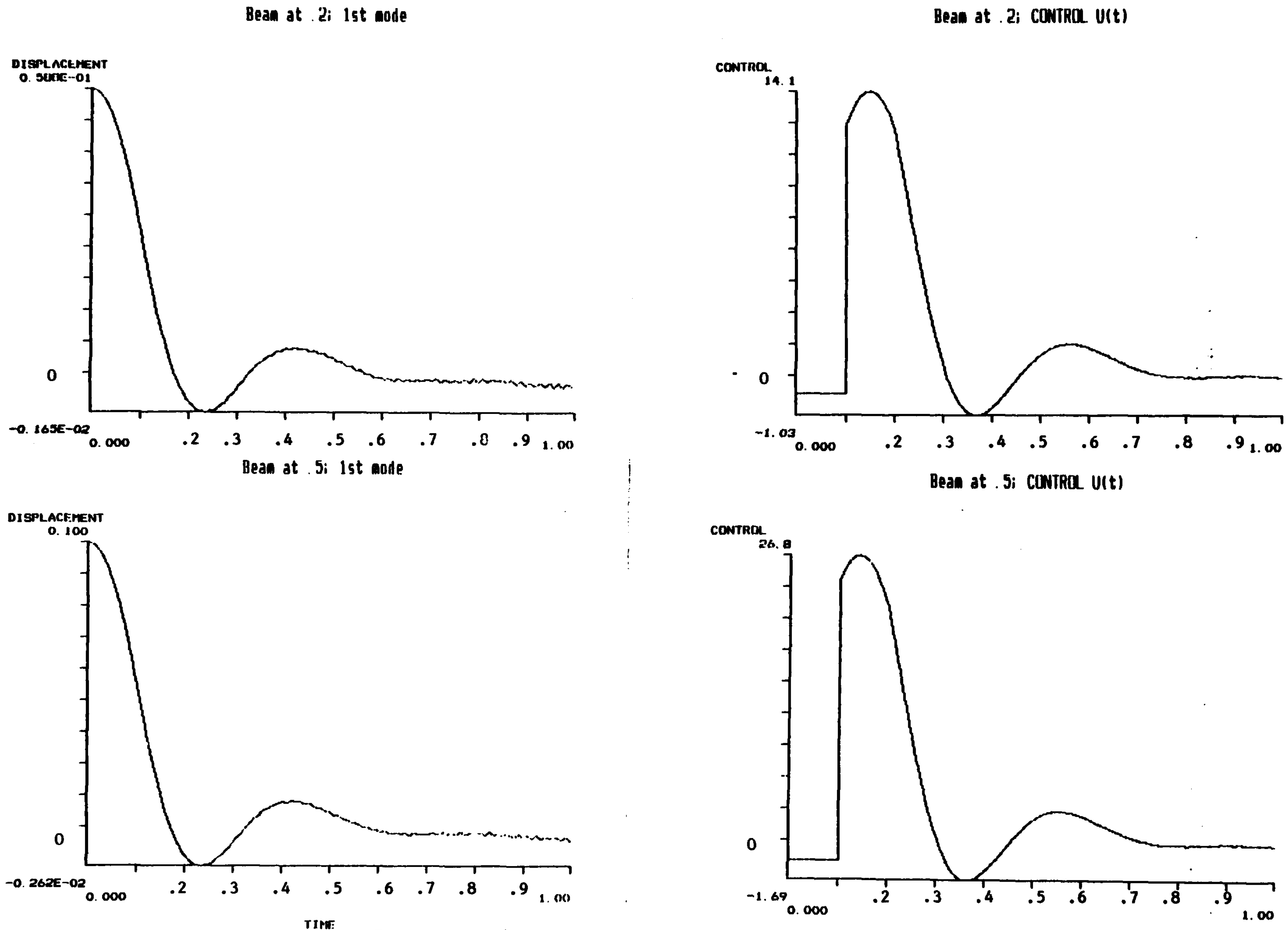


Three inputs/outputs at .2, .5, .8; delay = 0.1

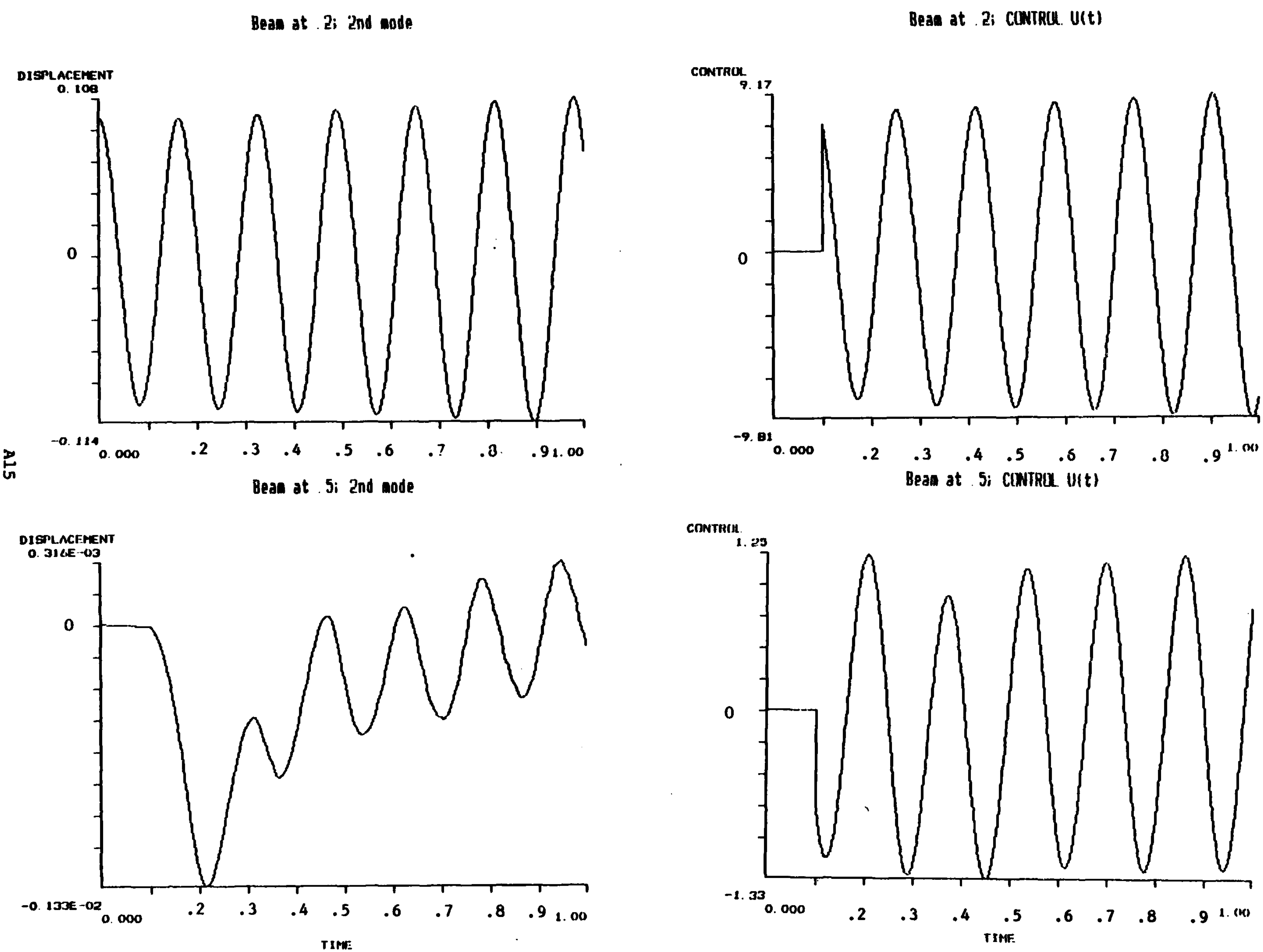




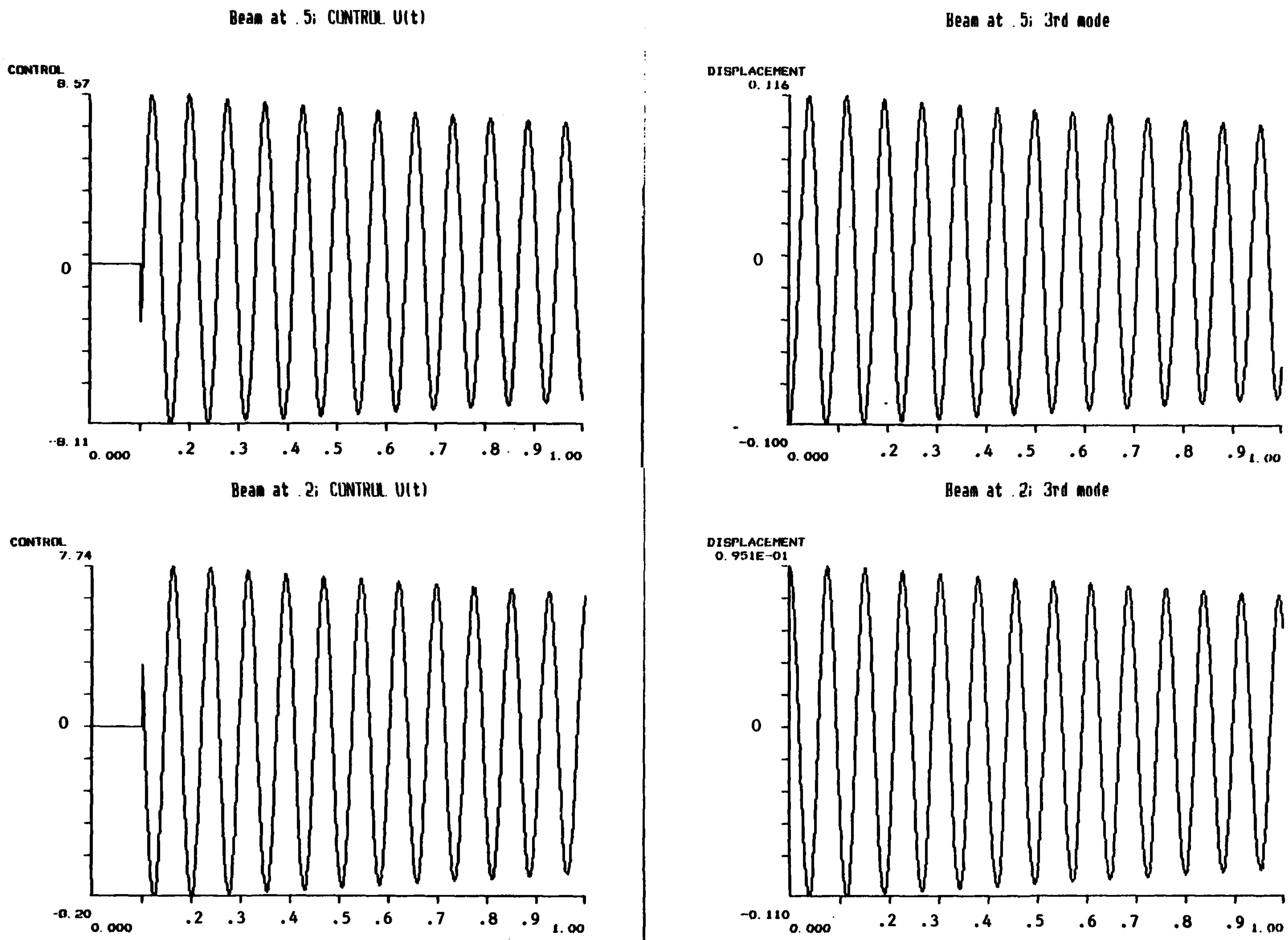
Three inputs/outputs at .2, .5 and .8; delay = 0.1



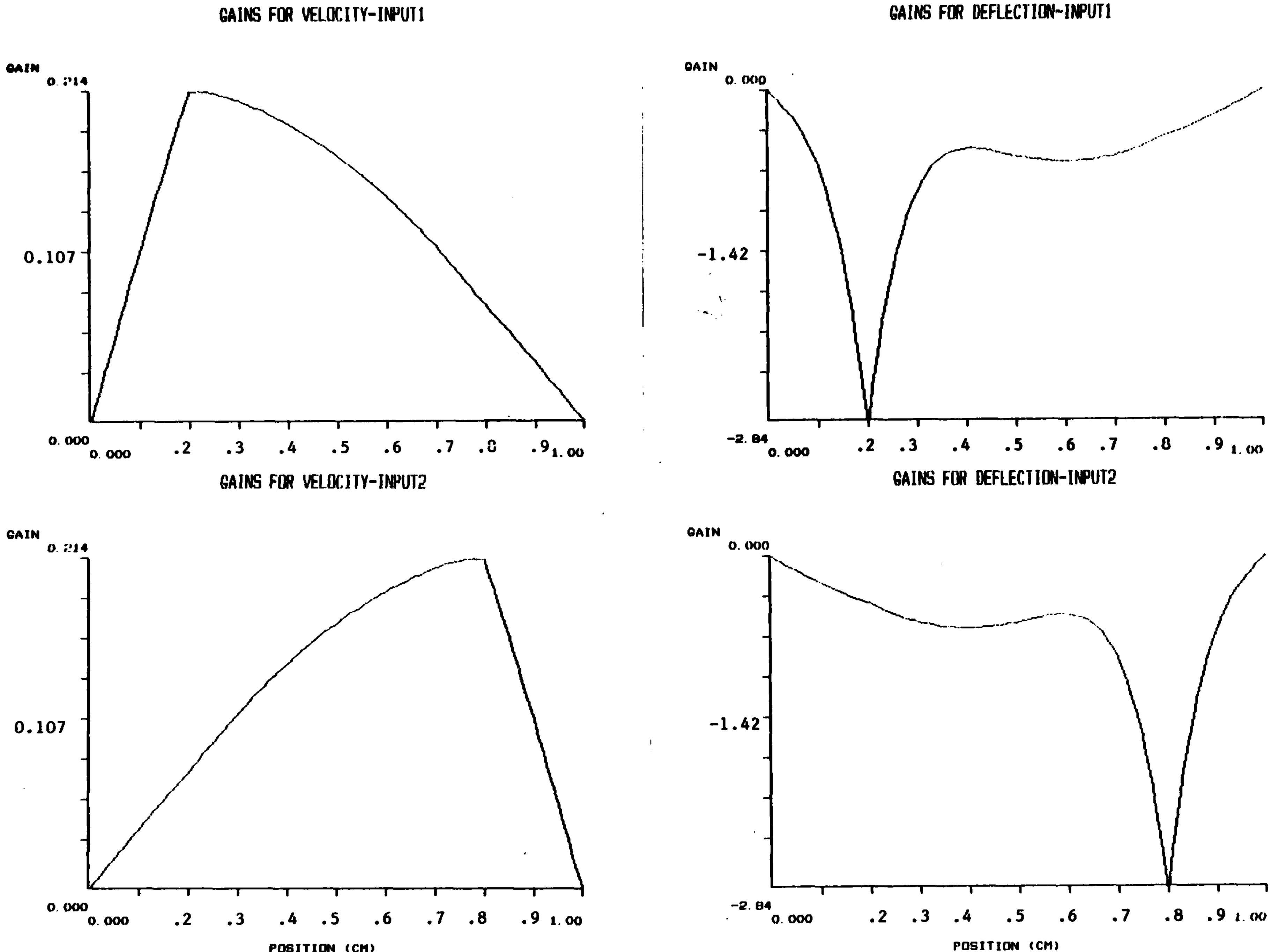
Three inputs/outputs at .2, .5, .8; delay = .1



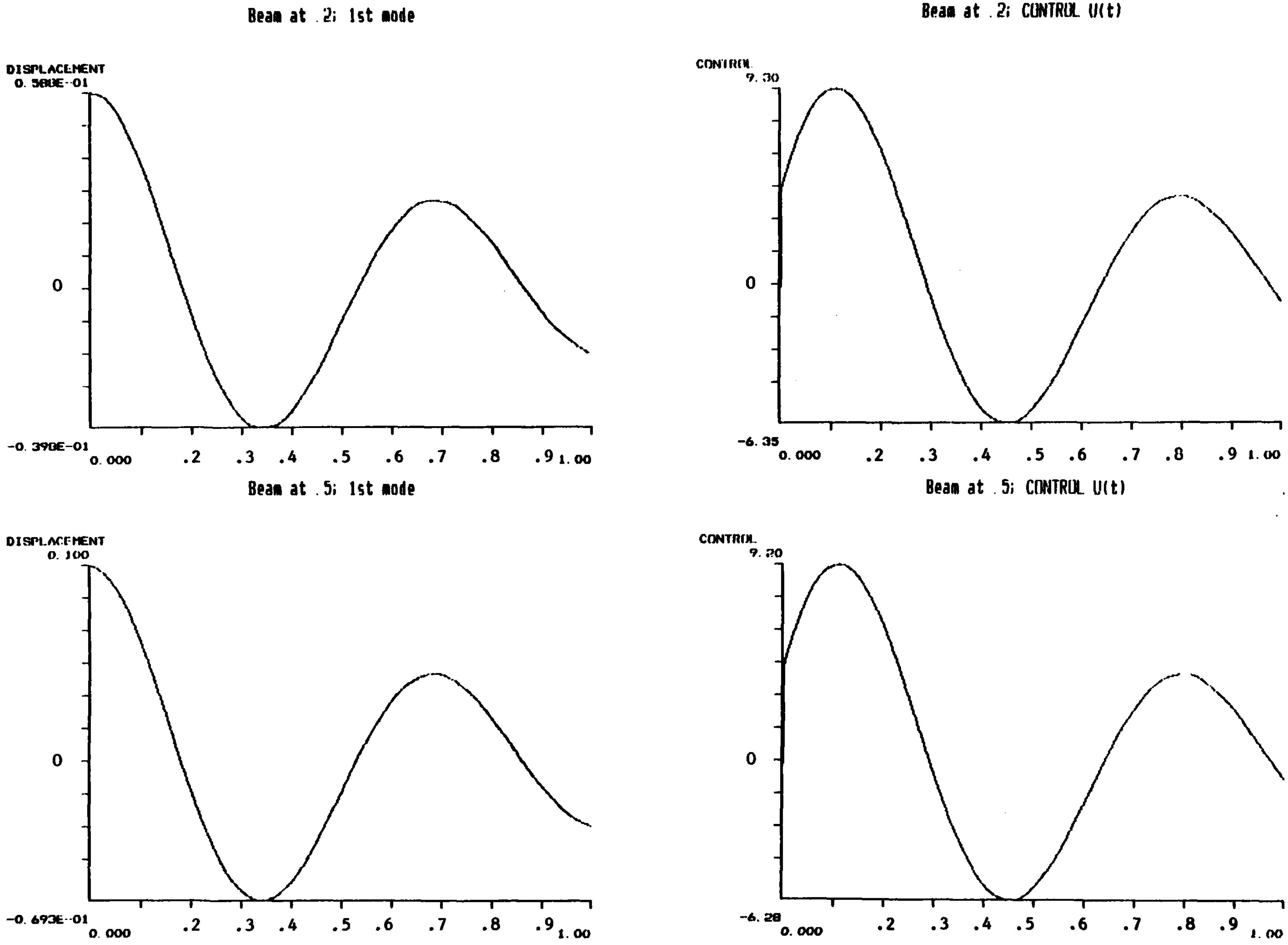
Three inputs/outputs at .2, .5, .8; delay = 0.1



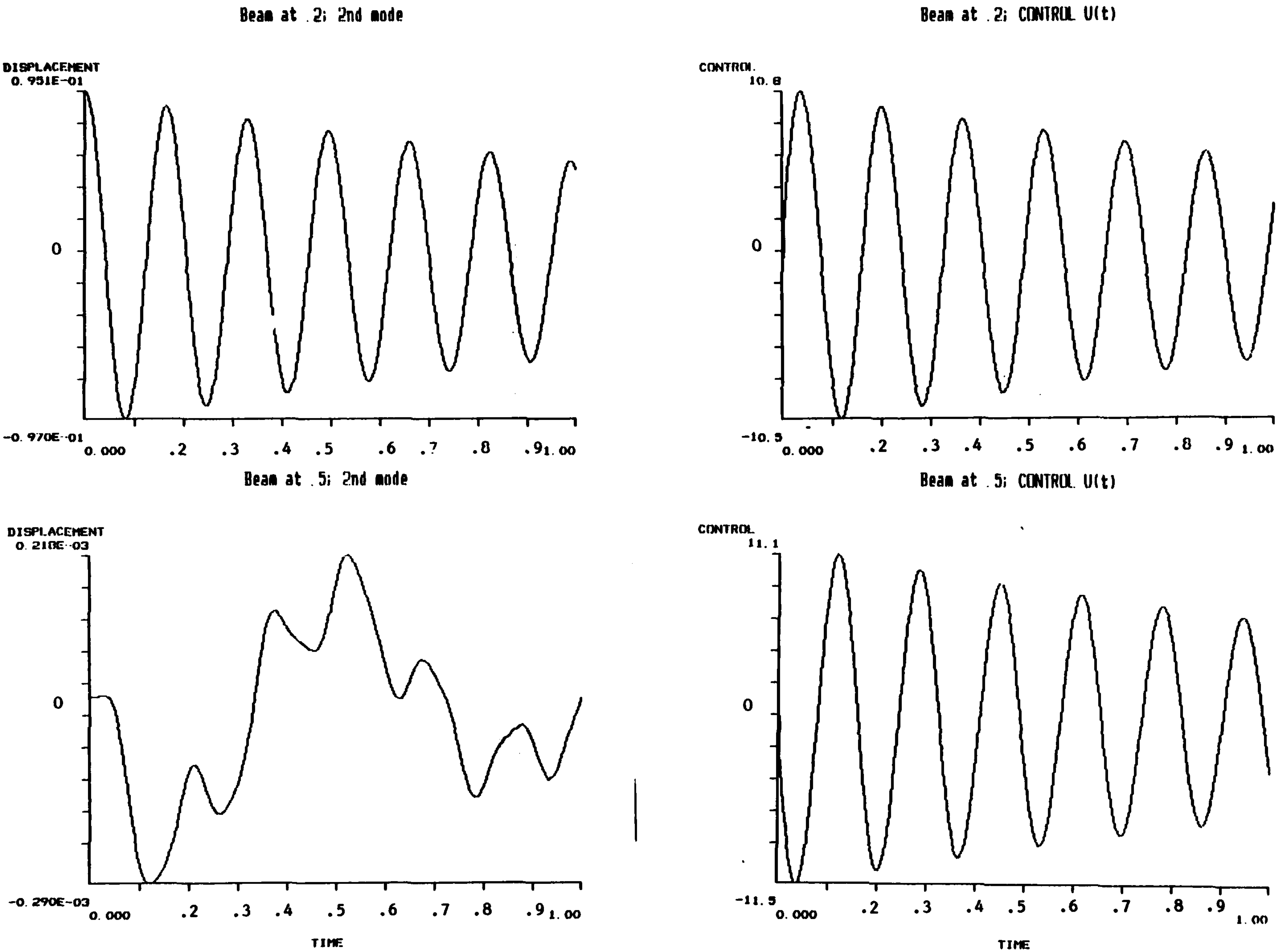
Three inputs/outputs at .2, .5, .8; delay = .1



Two inputs/outputs at .2, .8

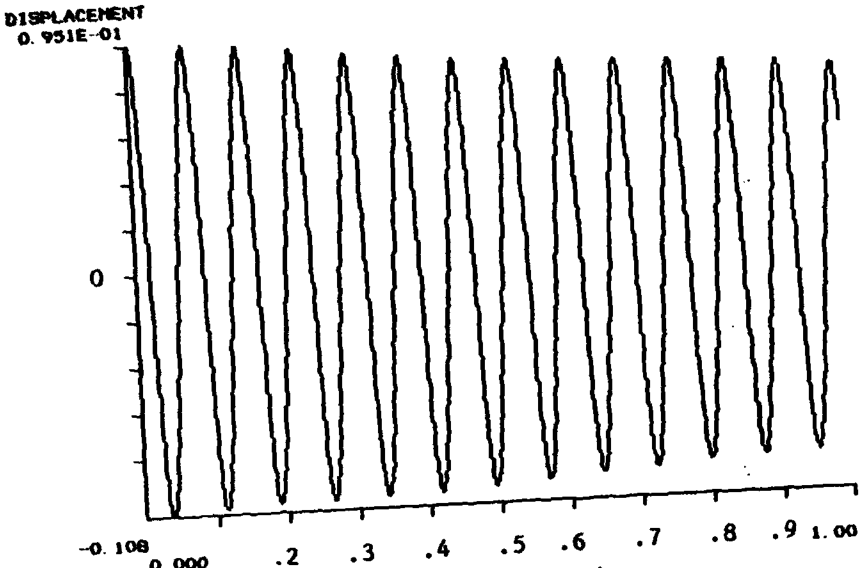


Two inputs/outputs at .2, .8

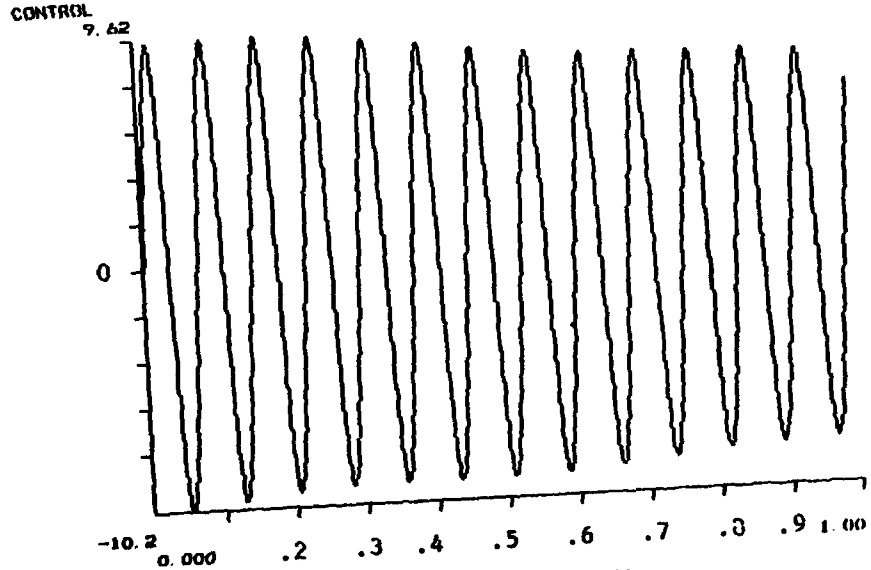


Two inputs/outputs at .2, .8

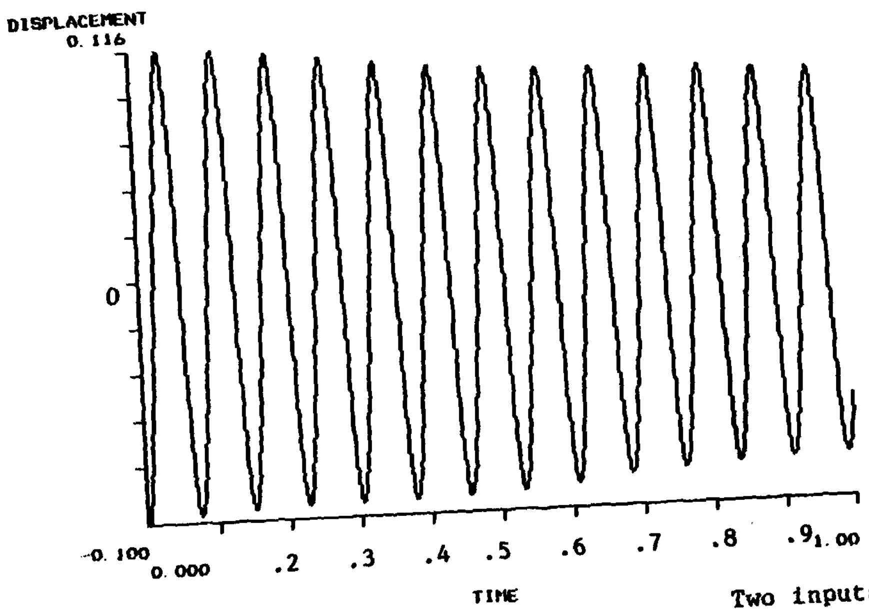
Beam at .2; 3rd mode



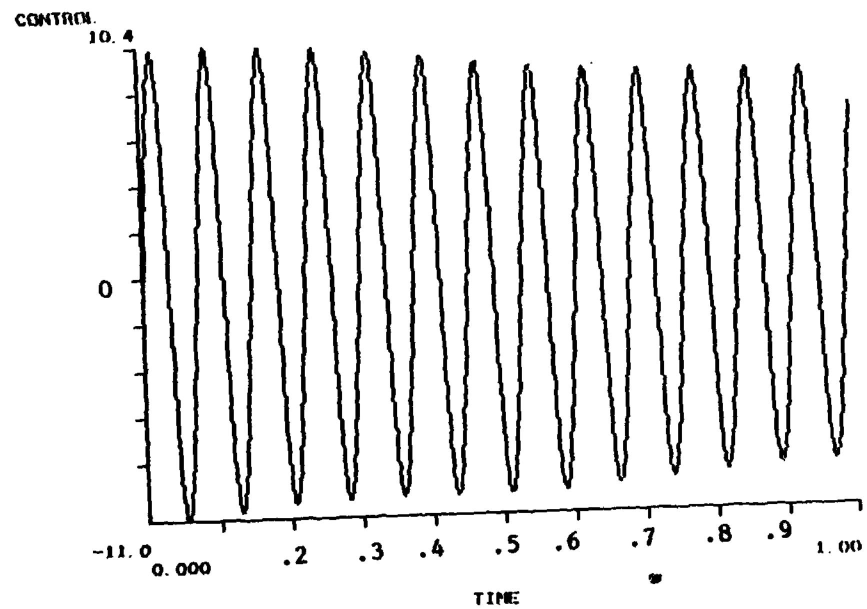
Beam at .2; CONTROL U(t)



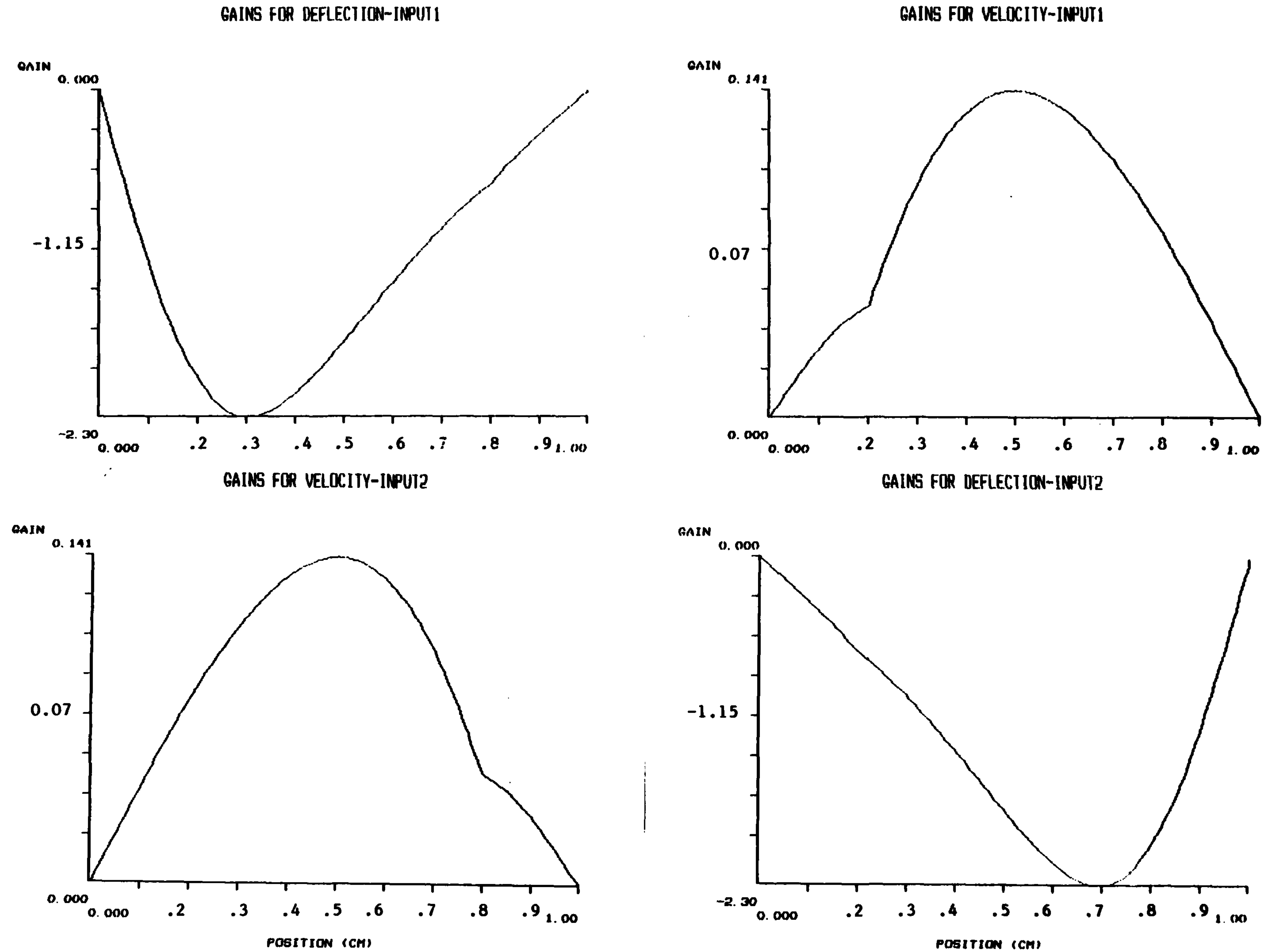
Beam at .5; 3rd mode



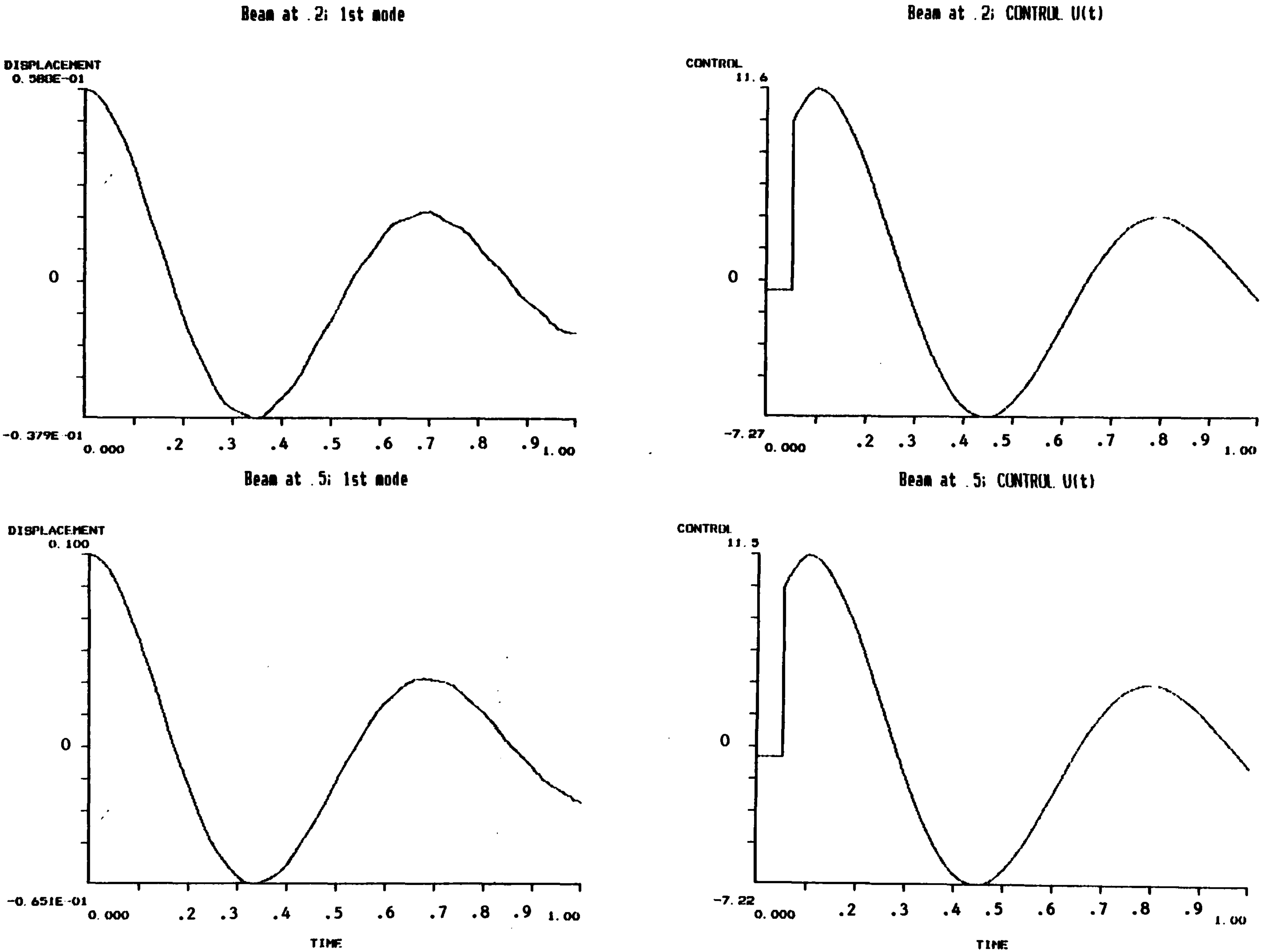
Beam at .5; CONTROL U(t)



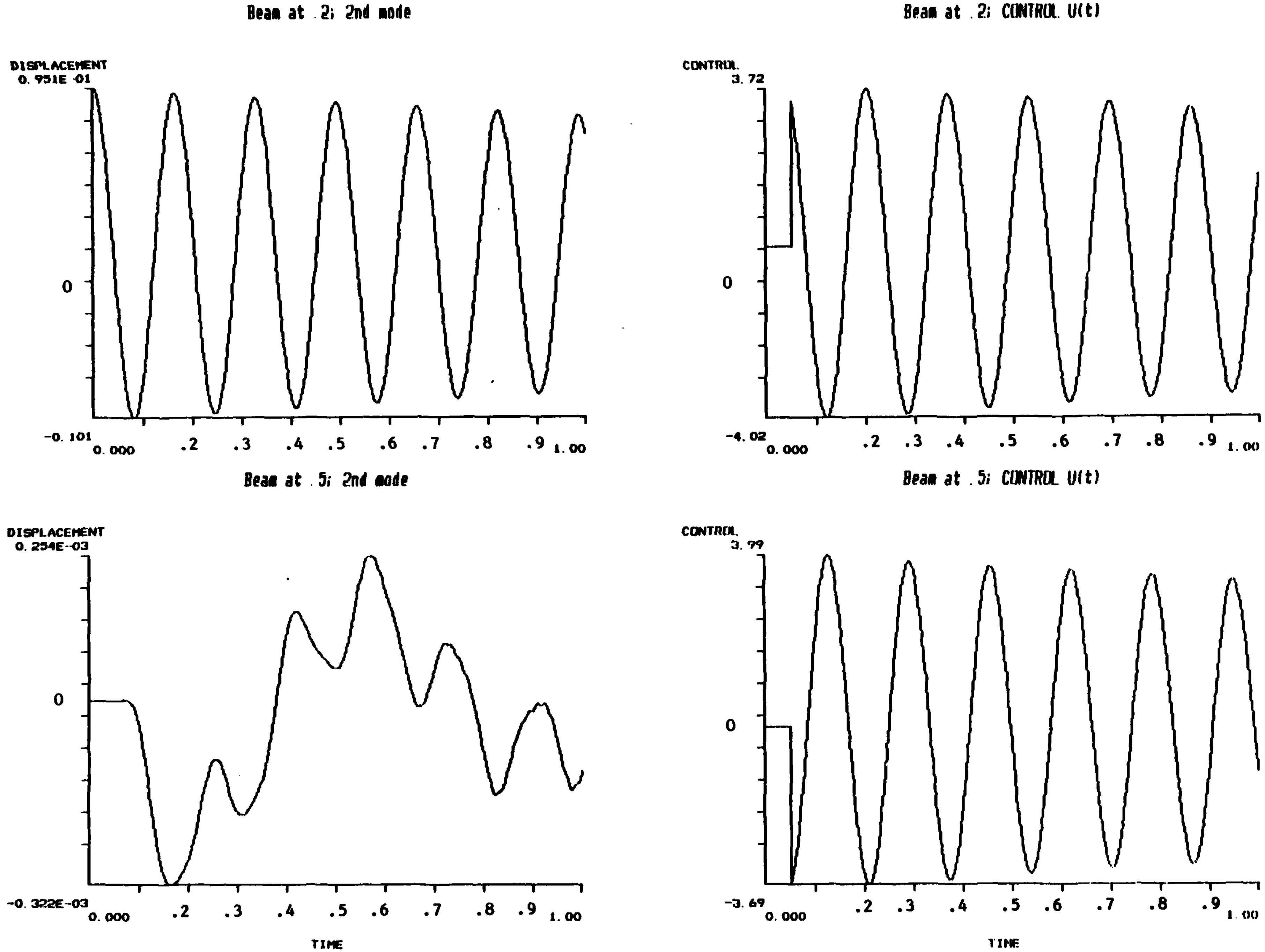
Two inputs/outputs at .2, .8



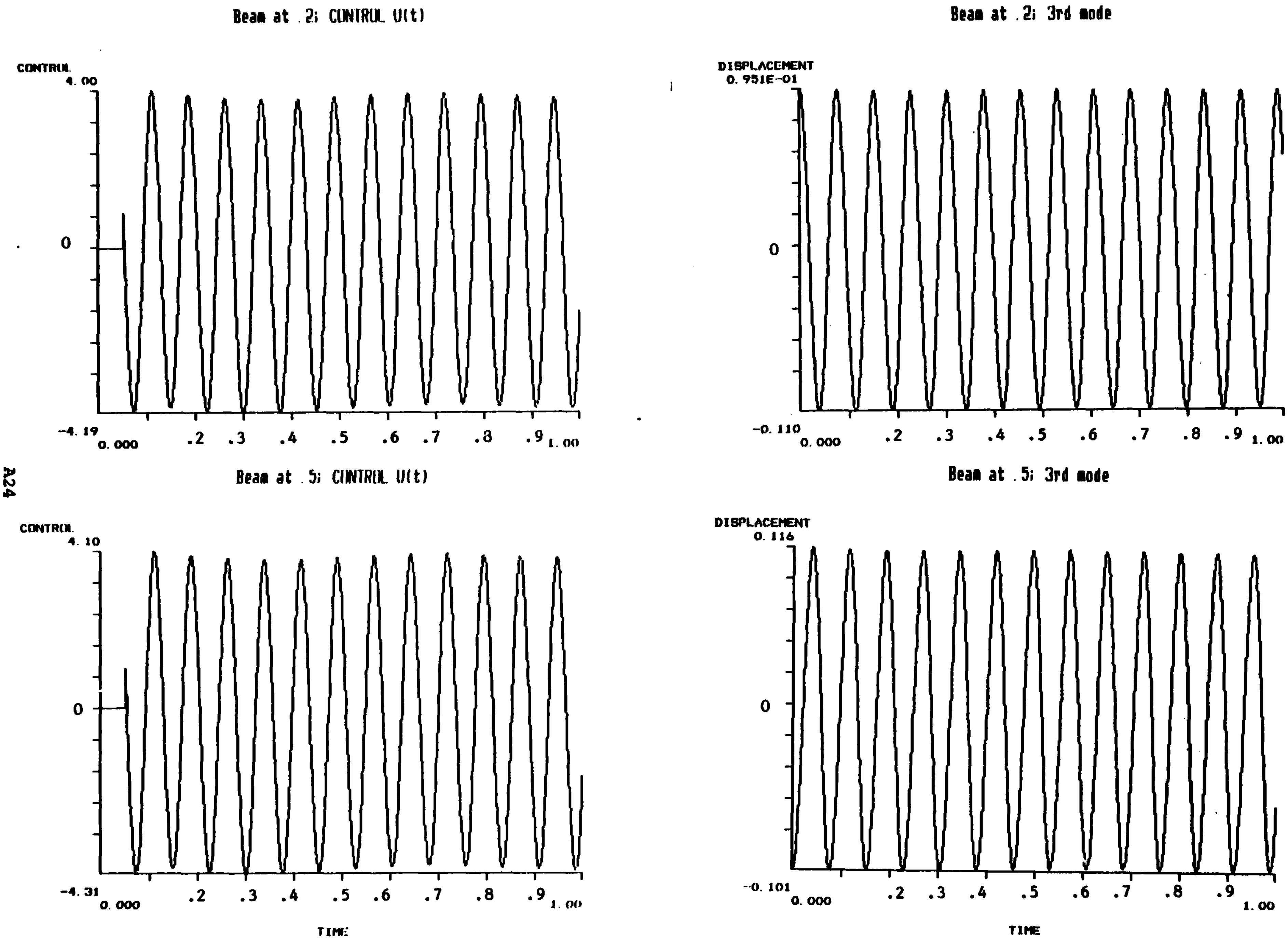
Two inputs/outputs at .2, .8; delay = .05



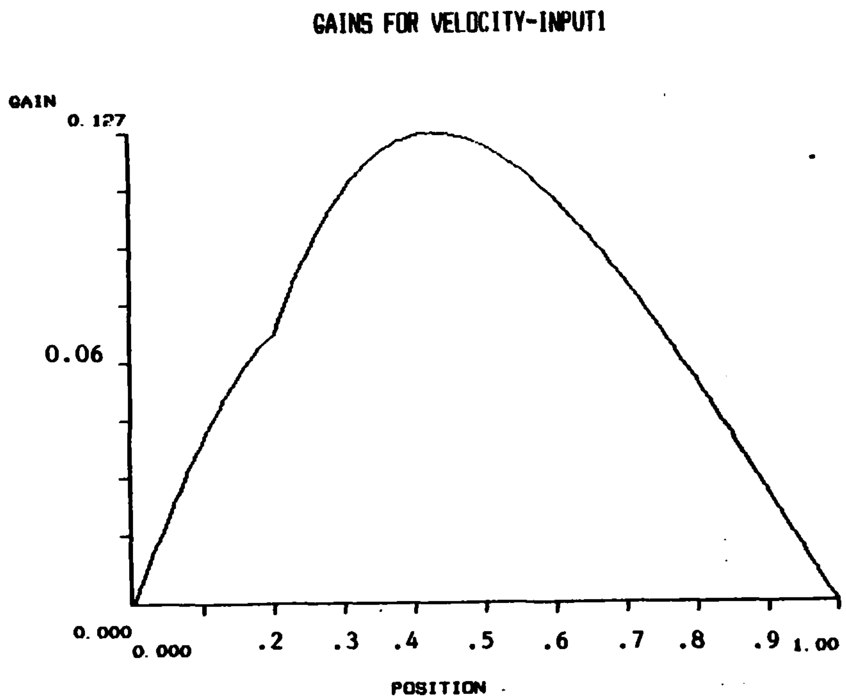
Two inputs/outputs at .2, .8; delay = 0.05

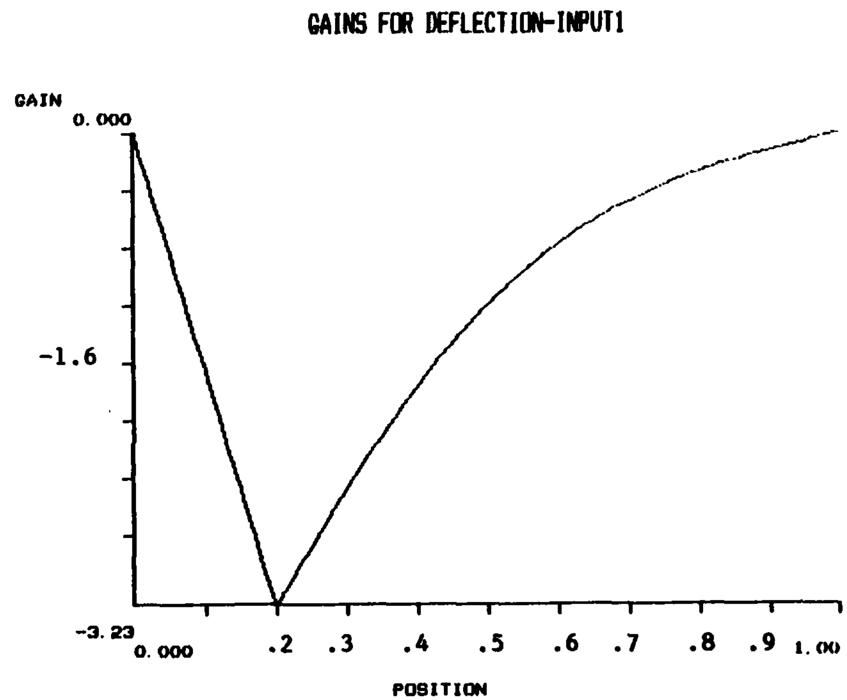


Two inputs/outputs at .2, .8; delay = 0.05



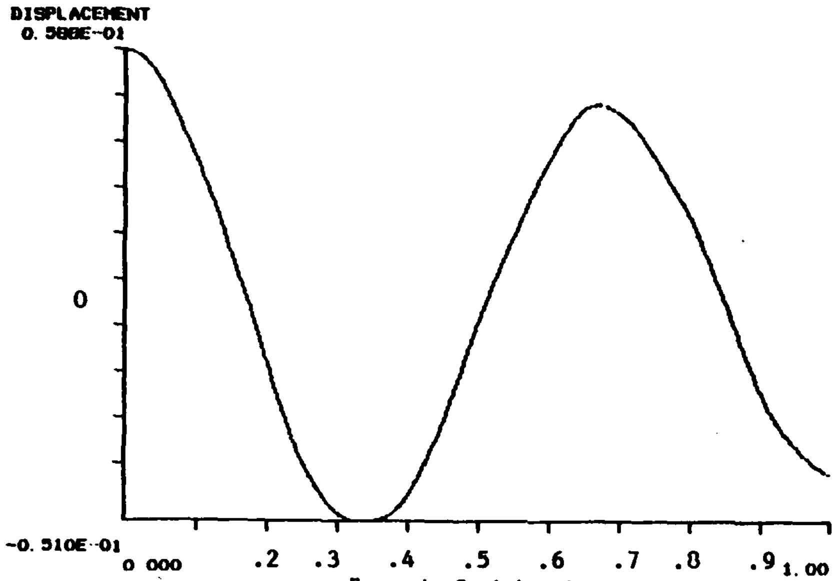
Two inputs/outputs at .2, .8; delay = 0.05



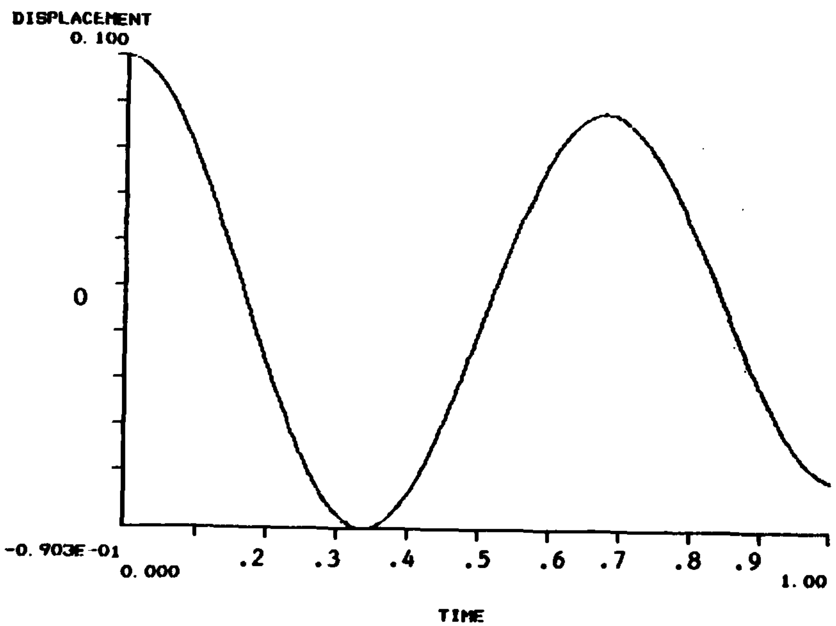


Input at .8; output at .2

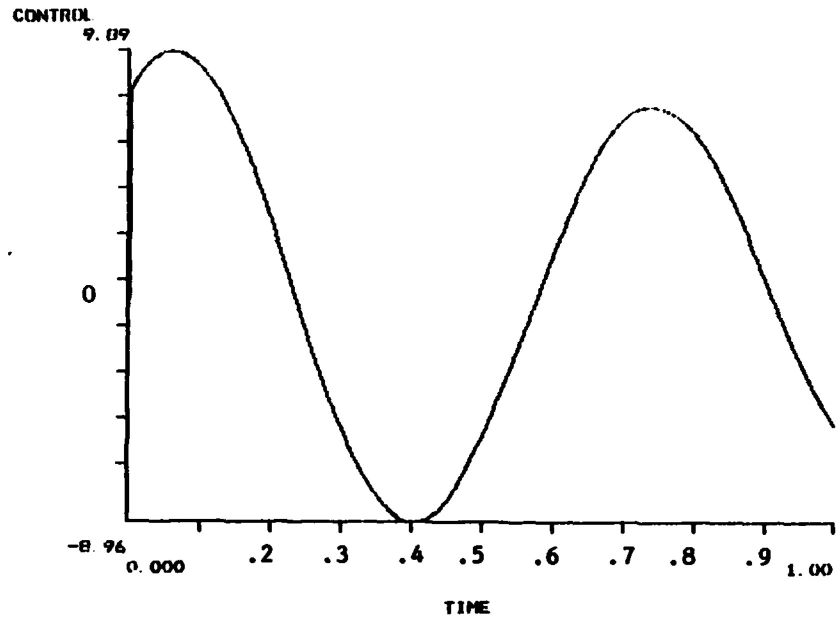
Beam at .2; 1st mode



Beam at .5; 1st mode

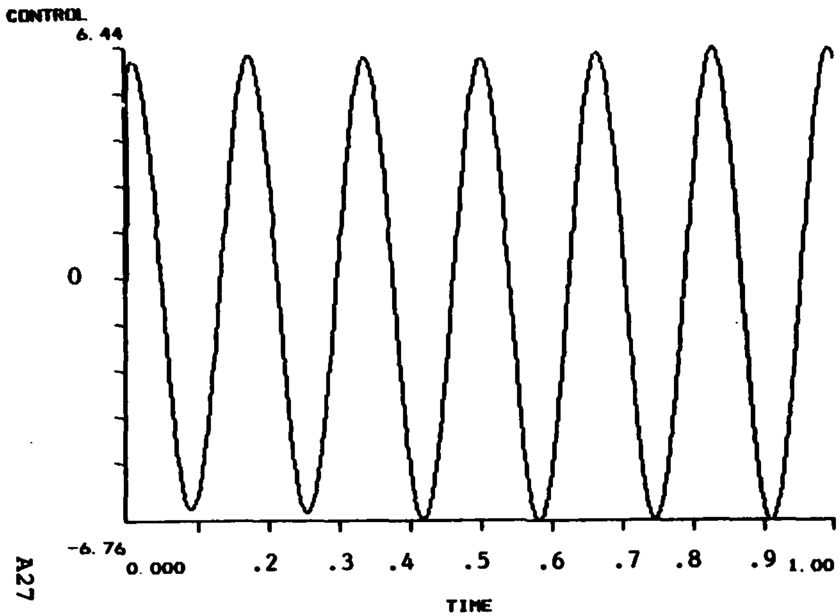


Beam at .2; CONTROL U(t)



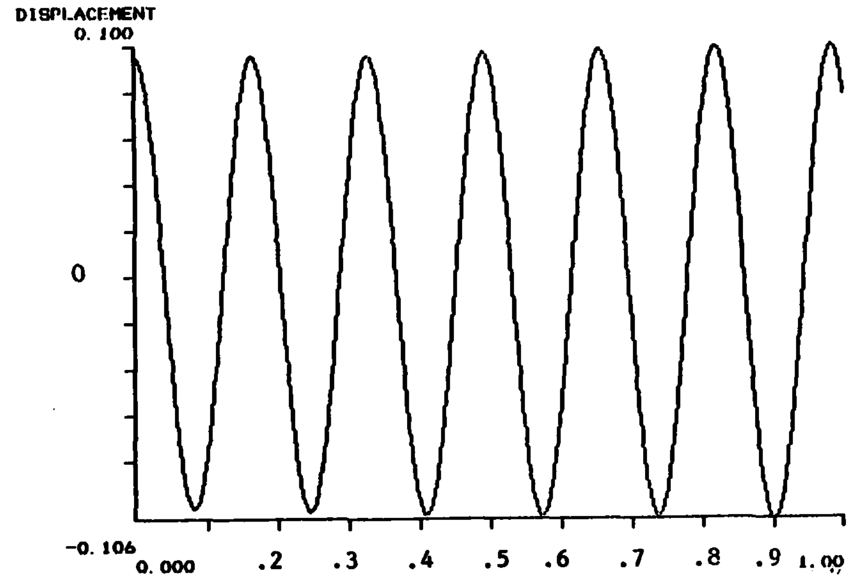
Input at .8, output at .2

Beam at .2; CONTROL U(t)

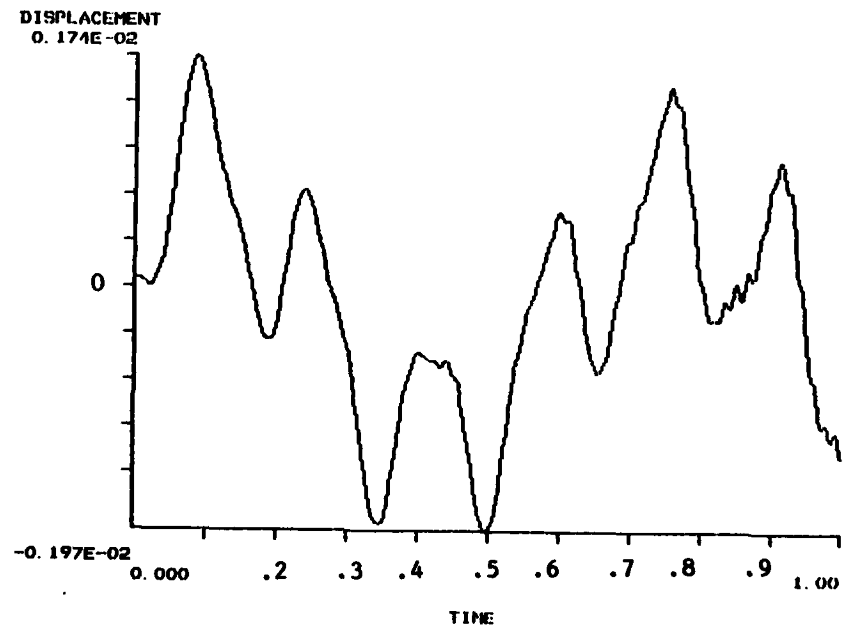


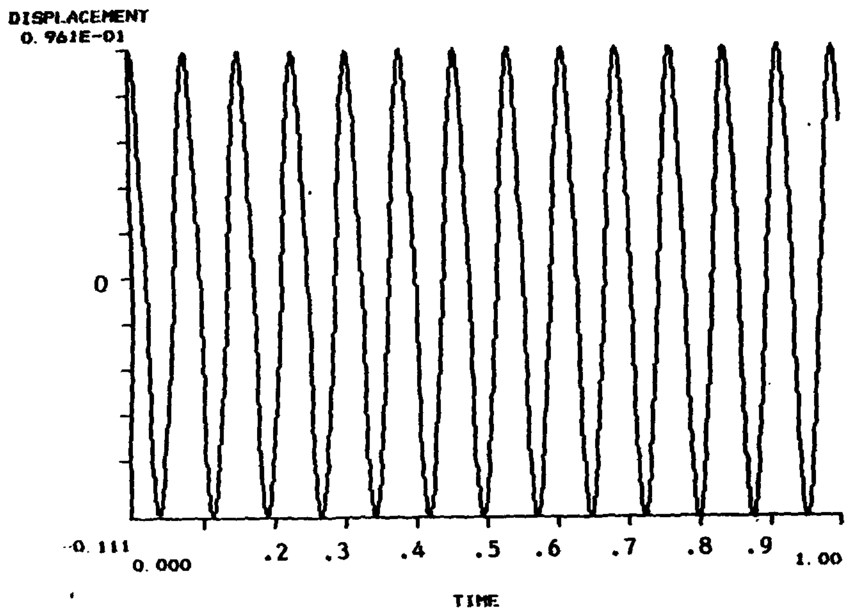
Input at .8, output at .2

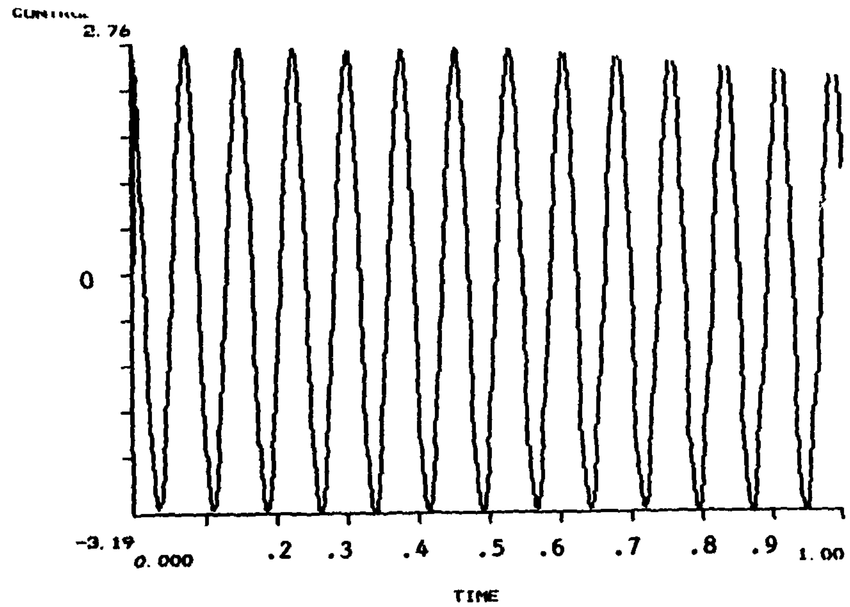
Beam at .2; 2nd mode



Beam at .5; 2nd mode







Input at .8, output at .2

END

FILMED

4-85

DTIC